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Multiscale Finite Element Methods for Flows on Rough Surfaces

Yalchin Efendiev[1], Juan Galvis[2,*] and M. Sebastian Pauletti[1]

[1] *Department of Mathematics, TAMU, College Station, TX 77843-3368, USA.*
[2] *Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia.*

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Abstract. In this paper, we present the Multiscale Finite Element Method (MsFEM) for problems on rough heterogeneous surfaces. We consider the diffusion equation on oscillatory surfaces. Our objective is to represent small-scale features of the solution via multiscale basis functions described on a coarse grid. This problem arises in many applications where processes occur on surfaces or thin layers. We present a unified multiscale finite element framework that entails the use of transformations that map the reference surface to the deformed surface. The main ingredients of MsFEM are (1) the construction of multiscale basis functions and (2) a global coupling of these basis functions. For the construction of multiscale basis functions, our approach uses the transformation of the reference surface to a deformed surface. On the deformed surface, multiscale basis functions are defined where reduced (1D) problems are solved along the edges of coarse-grid blocks to calculate nodal multiscale basis functions. Furthermore, these basis functions are transformed back to the reference configuration. We discuss the use of appropriate transformation operators that improve the accuracy of the method. The method has an optimal convergence if the transformed surface is smooth and the image of the coarse partition in the reference configuration forms a quasiuniform partition. In this paper, we consider such transformations based on harmonic coordinates (following H. Owhadi and L. Zhang [Comm. Pure and Applied Math., LX(2007), pp. 675–723]) and discuss gridding issues in the reference configuration. Numerical results are presented where we compare the MsFEM when two types of deformations are used for multiscale basis construction. The first deformation employs local information and the second deformation employs a global information. Our numerical results show that one can improve the accuracy of the simulations when a global information is used.

AMS subject classifications: 65N30, 58E20,76M50, 37E35

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*Corresponding author. *Email addresses:* efendiev@math.tamu.edu (Y. Efendiev), jcgalvisa@unal.edu.co (J. Galvis), pauletti@math.tamu.edu (M. S. Pauletti)

1 Introduction

Complex processes on rough surfaces occur in many applications. These include surface processes, such as diffusion on a rough terrains, or volume processes in geometrically complicated 3D thin regions. In addition, complex processes on rough surfaces can happen when the dominant heterogeneities form complex geometrical shapes. For example, if we consider the diffusion process in a heterogeneous media (i.e., coefficients representing conductivity are highly variable), then the diffusion in high-conductivity regions is a dominant factor that determines the outcome of these processes. One can write the diffusion equation restricted to the high conductivity region and approximate the resulting model by a diffusion equation on a rough surface. In summary, the small scales inherent to applications of diffusion problems on surfaces are caused by the presence of

- highly oscillatory geometrical properties;
- highly oscillatory conductivity coefficients.

Because of high spatial resolutions of these rough surfaces, the detailed simulations of complex processes can be prohibitively expensive. For this reason, some type of coarsening or upscaling is needed (see [3,39]). In these approaches, oscillatory geometric properties are represented on a coarse grid by local shape functions. These local shape functions are further coupled to solve the underlying problem on a coarse grid with a reduced computational cost. In this paper, we propose a new class of MsFEMs where the underlying fine-scale local problems are solved on a heterogeneous surface directly. In particular, we consider the problem of approximating the solution of the equation

$$-\operatorname{div}_s(\kappa\nabla_s u)=f \text{ in } \Gamma \quad \text{and} \quad u=g \text{ on } \partial\Gamma, \tag{1.1}$$

where ∇_s and div_s denote the surface gradient and the surface divergence, respectively.

In this paper, we construct Multiscale Finite Element Methods (MsFEMs) to approximate equation (1.1). We note that, MsFEMs are suited to obtain inexpensive approximations of problems with underlying complicated multiscale structures. MsFEMs consist of two major ingredients: (1) a small number of multiscale basis functions and (2) a global numerical formulation which couples these multiscale basis functions. Multiscale basis functions are designed to capture the effects on the solution caused by small scale parameters such as small scale geometrical variations of the domain where the problem is posed. In general, important small scale features of the solution need to be incorporated into these localized basis functions which contain information about the scales which are smaller (as well as larger) than the local numerical scale defined by the basis functions. In this paper, we study MsFEM approximation of multiscale elliptic problems on oscillatory surfaces.

We present a unified framework for MsFEMs that depends on a general coordinate transformation which deforms the reference surface and it is used to compute multiscale basis functions. This is motivated by the work of Owhadi and Zhang and suits well to