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On Direct and Semi-Direct Inverse of Stokes, Helmholtz and Laplacian Operators in View of Time-Stepper-Based Newton and Arnoldi Solvers in Incompressible CFD

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Abstract. Factorization of the incompressible Stokes operator linking pressure and velocity is revisited. The main purpose is to use the inverse of the Stokes operator with a large time step as a preconditioner for Newton and Arnoldi iterations applied to computation of steady three-dimensional flows and study of their stability. It is shown that the Stokes operator can be inversed within an acceptable computational effort. This inverse includes fast direct inverses of several Helmholtz operators and iterative inverse of the pressure matrix. It is shown, additionally, that fast direct solvers can be attractive for the inverse of the Helmholtz and Laplace operators on fine grids and at large Reynolds numbers, as well as for other problems where convergence of iterative methods slows down. Implementation of the Stokes operator inverse to time-stepping-based formulation of the Newton and Arnoldi iterations is discussed.

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1 Introduction

This study is motivated by many successful applications of an inverse Stokes operator as preconditioners for steady state Newton solvers and Arnoldi eigensolvers [1,2]. The Stokes operator inverse is considered as an intrinsic part of a pressure/velocity coupled time-dependent Navier-Stokes solver, which connects between time-dependent calculations and direct numerical solution for steady states and their stability. For examples of

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a successful use of this technique and results on multiplicity and stability of various flow states we refer to [3–6] and references therein. An extension of this approach for calculation of leading eigenvalues in a prescribed frequency range is proposed in the recent paper [7]. A further extension to study of non-modal optimal disturbances growth is described in [8].

To become an effective preconditioner and to be applied as in [1–8] to large Reynolds number flows, the Stokes operator must be evaluated with a large time step. The latter becomes especially difficult when three-dimensional flows are studied on fine grids making most of traditional iterative methods to converge unacceptably slowly. In particular, we are interested in coupled incompressible pressure-velocity solvers, which are more computationally demanding than segregated ones, but possess important advantages: more stable time integration, correct calculation of pressure at each time step, and a possibility to proceed without pressure boundary conditions. Applied as preconditioners to Newton and Arnoldi solvers the coupled methods are expected to perform well if the Stokes operator with a large time step can be inversed within a relatively short CPU time. Considering 2D stability problems one can apply a direct sparse solver to inverse the 2D Stokes operator [9], however this becomes too memory demanding for fine three-dimensional grids. A similar approach with the same restrictions in 3D cases was implemented in [10] for explicitly calculated Jacobian matrices. At the same time, our recent pressure-velocity coupled multigrid solver [11], which performs well at small time steps fails to converge at large steps needed for 3D stability studies [10,11]. Based on the above experience, in this paper we recall the well-known factorization of the Stokes operator, which we use for computation of its inverse, applying fast direct methods where possible. Using the finite volume method, we arrive to an analog of the Uzawa scheme [12], in which only one matrix, called "pressure matrix" has to be inversed iteratively. As a result, we arrive to a time-stepping method, which may be too CPU-time consuming for a straight-forward time-integration, but yields the inverse of the Stokes operator with a large time step, for which we are seeking.

Another important observation of this study relates to fully three-dimensional time-dependent CFD modelling at very large Reynolds numbers, where all the known iterative methods slow down or fail. Here we observe that for calculation on fine grids and at large Reynolds numbers the eigenvalue decomposition based direct solver [13] becomes more efficient than iterative solvers. Since computational requirements of the direct solver do not depend on the time step and Reynolds number, all the time steps are completed within the same CPU time, which is an attractive feature by itself. This observation is not completely new, but the fact still is not widely recognized. We show also that the eigenvalue decomposition based direct solver allows for an efficient parallelization in a distributed memory multiprocessor computer.

As a preliminary step, we examine computational performance of the above direct solver when implemented in a pressure-velocity segregated time-integration solver. We consider a series of well-known natural convection benchmarks for the purpose of comparing performance of the direct solvers with that of an iterative method. We have chosen