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A Fast Accurate Boundary Integral Method for Potentials on Closely Packed Cells

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Abstract. Boundary integral methods are naturally suited for the computation of harmonic functions on a region having inclusions or cells with different material properties. However, accuracy deteriorates when the cell boundaries are close to each other. We present a boundary integral method in two dimensions which is specially designed to maintain second order accuracy even if boundaries are arbitrarily close. The method uses a regularization of the integral kernel which admits analytically determined corrections to maintain accuracy. For boundaries with many components we use the fast multipole method for efficient summation. We compute electric potentials on a domain with cells whose conductivity differs from that of the surrounding medium. We first solve an integral equation for a source term on the cell interfaces and then find values of the potential near the interfaces via integrals. Finally we use a Poisson solver to extend the potential to a regular grid covering the entire region. A number of examples are presented. We demonstrate that increased refinement is not needed to maintain accuracy as interfaces become very close.

AMS subject classifications: 35J05, 65N06, 65N38

Key words: Laplace equation, boundary integral method, fast multipole method.

1 Introduction

A wide range of biological problems lead to models involving a potential function in tissue with a number of closely packed cells. Recent applications include gene transfection [14, 15], electrochemotherapy of tumors [20] and cardiac defibrillation [1]. Our interest in the problem is mainly motivated by studies of the electrical response of biological cells under field stimulation [25], which can be described by harmonic potential

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functions on a domain consisting of many cells and an extracellular region with different conductivities. These potentials are naturally expressed as integrals on the cell boundaries. Boundary integral methods are well suited for the computation of such problems, but they require special care when the cell boundaries are close. We present a method which is designed to handle such cases accurately and efficiently. We focus here on the model of electrical stimulation of cells in two dimensions, but similar computational issues occur in other applications. For example, the motion of drops of one viscous fluid in another, or the fluid motion of vesicles, such as blood cells, is often modeled by Stokes flow, leading to a related integral formulation, again with many components embedded in a surrounding medium [16, 22, 23, 26, 27].

The electric potential problem is formulated in Sections 2 and 3. Since the potential is a harmonic function inside each cell and in the exterior region, with prescribed boundary conditions at the cell interfaces, it is natural to write the potential as a sum of single and double layer potentials on the cell boundaries Γ_k, $k=1,\cdots,K$, and evaluate the integrals directly. In principle this is routine if the point of evaluation $\mathbf{x}$ is away from Γ_k. It is also not difficult in this two-dimensional setting if $\mathbf{x}\in\Gamma_k$. However, if, for example, Γ_1 and Γ_2 are close and $\mathbf{x}\in\Gamma_1$, then the integral on Γ_2 is nearly singular, so that a standard quadrature rule becomes inaccurate when the distance is small. It is therefore desirable to use a method of quadrature which is accurate, uniformly with respect to the point of evaluation, without requiring a large amount of extra work. A method with these features was developed in [5] and is used here. Briefly, the singularity in the integral kernel is regularized on a scale comparable to the grid size, and a standard quadrature is used for the regularized integral. Analytical corrections are then added for the errors due to regularization and discretization. It is not necessary to use special quadrature points depending on the point of evaluation; the method is almost as efficient as for a smooth integrand. The present work makes practical use of this integration method in the case where several interfaces are close to each other. The quadrature and correction formulas are given in detail in Section 4. A similar method for layer potentials on surfaces was developed in [4].

To solve for the potential, we first solve an integral equation for a source term on the cell boundaries. We then compute the potential at grid points covering the region of interest. To compute the integrals directly would require a large computational cost, especially if the number of cells is large. For this reason we use a version of the fast multipole method for the interaction of points which are well separated; the effect of the regularization can be neglected in this case. The nearby interactions are summed directly, using the regularized kernel. This procedure is explained in Section 5. The regularization must be used within a large enough radius for accuracy, but it imposes a lower bound on the capacity in the tree structure for the fast summation. Guidelines for the choice of parameters are given to balance the accuracy and efficiency. A similar procedure was used in the context of regularized point vortices in [10].

After solving the integral equation, we evaluate the potential at grid points near the cell boundaries. These values are again given by nearly singular integrals, which are