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A Sylvester-Based IMEX Method via Differentiation Matrices for Solving Nonlinear Parabolic Equations

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Abstract. In this paper we describe a new pseudo-spectral method to solve numerically two and three-dimensional nonlinear diffusion equations over unbounded domains, taking Hermite functions, sinc functions, and rational Chebyshev polynomials as basis functions. The idea is to discretize the equations by means of differentiation matrices and to relate them to Sylvester-type equations by means of a fourth-order implicit-explicit scheme, being of particular interest the treatment of three-dimensional Sylvester equations that we make. The resulting method is easy to understand and express, and can be implemented in a transparent way by means of a few lines of code. We test numerically the three choices of basis functions, showing the convenience of this new approach, especially when rational Chebyshev polynomials are considered.

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1 Introduction

One of the most remarkable properties that distinguish nonlinear evolution problems from linear ones is the possibility of an eventual occurrence of singularities in a finite time T, starting from perfectly smooth data. One of the simplest forms of spontaneous singularities in nonlinear problems appears when one or more of the dependent variables

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tend to infinity as $t\to T$, where $T\in(0,\infty)$ is called the blow-up time. A singularity represents often an important change in the properties of the models, which explains why it is important to analyze them and to reproduce them accurately by a numerical method.

In this paper, we consider the classical semi-linear parabolic equation

$$u_t=\Delta u+u^p,\qquad p>1,\quad \mathbf{x}\in\mathbb{R}^n,\quad t>0, \tag{1.1}$$

with initial condition

$$u(\mathbf{x},0)=u_0(\mathbf{x}), \tag{1.2}$$

where $u_0(\mathbf{x})$ is continuous, non-negative and bounded.

The local existence in time of positive solutions of (1.1)-(1.2) follows from standard results, but the solution may develop singularities in finite time. More precisely, we have the following theorem [13,23]:

Theorem 1.1. *Let* $p_c(n)=1+\frac{2}{n}$.

1. *If* $1<p<p_c(n)$, *for any non-trivial solution of* (1.1)-(1.2), *there exists a finite time* T, *such that*
$$\limsup_{t\nearrow T}\left(\sup_{\mathbf{x}\in\mathbb{R}^n}u(\mathbf{x},t)\right)=+\infty.$$
2. *If* $p>p_c(n)$, *there exists a global positive solution, if the initial values are sufficiently small (less than a small Gaussian).*

In the first case, we say that $u(x,t)$ blows up in a finite time T, which is called the blow-up time of u; and p_c is the critical exponent of the problem.

Another important question is to determine the asymptotic behavior of the solution, as the blow-up is approached. There are several references on this topic: in [18], the different possible asymptotic behaviors for $n=1$ are described, and in [33], the case $n>1$ is studied.

In the following pages, we will develop and test a new matrix-based pseudo-spectral method for (1.1) in two and three spatial dimensions. References on spectral methods can be found in [5,7,11,29,31,32], together with the more classical [6,15]. One of the main difficulties of dealing with (1.1) is the unboundedness of the spatial domain; nevertheless, according to Boyd [5, p. 338], the many options for unbounded domains fall into three broad categories:

1. Domain truncation (approximation of $x\in\mathbb{R}$ by $[-L,L]$, with $L\gg1$);
2. Basis functions intrinsic to an infinite interval (Hermite functions, sinc functions);
3. Mapping of the unbounded interval to a finite one, followed by application of Chebyshev polynomials or a Fourier series.