Nearly Singular Integrals in 3D Stokes Flow

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Abstract. A straightforward method is presented for computing three-dimensional Stokes flow, due to forces on a surface, with high accuracy at points near the surface. The flow quantities are written as boundary integrals using the free-space Green's function. To evaluate the integrals near the boundary, the singular kernels are regularized and a simple quadrature is applied in coordinate charts. High order accuracy is obtained by adding special corrections for the regularization and discretization errors, derived here using local asymptotic analysis. Numerical tests demonstrate the uniform convergence rates of the method.

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1 Introduction

Low Reynolds number flows are fundamental in a large class of problems, for example, particle and drop motion, the swimming of microorganisms, vesicle flows [8, 15, 20, 22]. These phenomena are modeled by the Stokes equations, and a wide variety of numerical techniques have been employed to find solutions, among which boundary integral equation and singularity based methods are most popular. Boundary integral equation methods have several well-known advantages, such as reduction in the dimensionality of the problem and high achievable accuracy of the solution. They have been used effectively to simulate the behavior of drops or vesicles in Stokes flow; comprehensive work includes [19, 22, 25].

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The numerical treatment of integrals similar to the Stokes problem is studied extensively in many works. The discretization of integral equations in three dimensions is usually based on Galerkin or collocation methods, where basis functions are defined and an integration method is used to construct the matrix coefficients. This often involves a product integration rule or a change of variables. High accuracy solutions can be obtained at points on the boundary and far away from it. When the solution is evaluated on dense grids, the integrals become nearly singular if the evaluation point is close to the surface. This issue is often overlooked; the few works that address it include [2,3,10,12,23]. In [4], partitions of unity were used along with an analytical resolution of the singularity by a change to polar coordinates, in the context of surface scattering problems. These local quadrature methods were extended in [23] to various elliptic problems; for points close to the boundary, interpolation from the solution at far away points was used to achieve higher accuracy.

In Stokes flow simulations, the boundaries, e.g., two drops, often get close to each other, and computing values at nearby points accurately becomes a non-trivial problem. The method of regularization and correction of [2,3] is well suited to handle this difficulty, since it is simple to implement and the work needed does not increase with proximity to the surface. In [2], harmonic functions written as single and double layer potentials on curves were computed with uniformly high accuracy with respect to the evaluation point. The free-space Green's function for the Laplacian is regularized using a small parameter, and the integrals are discretized by a simple quadrature rule. Asymptotic analysis for the integral near the singularity leads to closed form expressions for the leading terms in error due to regularization, and also due to the discretization quadrature. These expressions are then added as corrections to yield higher accuracy of the numerical solution. This integration technique was applied to 2D Stokes flow with a moving elastic interface in [14]. Related integration formulas for double layer potentials for the Laplacian on closed surfaces in 3D were derived in [3]. Two-dimensional boundary integral calculations for a scalar problem from electromagnetics with several boundaries close to each other were done in [24]. In [16], 3D doubly periodic electromagnetic scattering was computed using regularization and corrections for points on the surface.

In this paper, we overcome the issue of near singularity in Stokes flow by extending the method of [3] to evaluate the integrals for velocity and pressure due to forces on closed surfaces in three dimensions. The surface is represented by several overlapping patches, each parametrized in a rectangular system. The method is based on a direct, or Nyström, discretization and partitions of unity, where the Green's function for the Laplacian is regularized. To evaluate the integrals near the surface, correction terms for the error are added to achieve high accuracy; these corrections are derived here and in [3] using local asymptotic analysis. The corrections are local and therefore the high order convergence is achieved without increasing the overall computational complexity. Another important aspect of this method is that the integrals are computed with regularly spaced quadrature points, without special gridding or cut-off near the singularity as in [4,23], and the spacing does not change when the evaluation point is close to the