Commun. Comput. Phys.
doi: 10.4208/cicp.210611.240212a

Vol. **13**, No. 2, pp. 361-385
February 2013

Flow in Collapsible Tubes with Discontinuous Mechanical Properties: Mathematical Model and Exact Solutions

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Received 21 June 2011; Accepted (in revised version) 24 February 2012

Communicated by Chi-Wang Shu

Available online 28 June 2012

Abstract. We formulate a one-dimensional time-dependent non-linear mathematical model for some types of physiological fluid flow in collapsible tubes with discontinuous material properties. The resulting 6×6 hyperbolic system is analysed and the associated Riemann problem is solved exactly. Although the solution algorithm deals with idealised cases, it is nonetheless uniquely well-suited for assessing the performance of numerical methods intended for simulating more general situations. Moreover, our model may be a useful starting point for numerical calculations of realistic flows involving rapid and discontinuous material property variations. One important example in mind is the simulation of blood flow in medium-to-large veins in humans. Finally, we also discuss some peculiarities of the model regarding the loss of strict hyperbolicity and uniqueness. In particular we show an example in which the solution of the Riemann problem is non unique.

AMS subject classifications: 76Z05

Key words: Collapsible tubes, physiological flows, blood flows, Riemann problem.

1 Introduction

The theoretical study of flow phenomena in humans through mathematical models is closely related to the study of flow of an incompressible liquid in thin-walled collapsible tubes. In fact the applicability of theoretical models for thin-walled collapsible tubes covers a wider variety of physiological phenomena as well the design of clinical devises for practical medical applications. Fluid flow through compliant tubes is usually used

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to represent physiological flows such as blood flow in arteries and veins, air flow in the airways and urine flow in the ureter. In this paper we are interested in theoretical models for determining flow patterns and the geometry of the tube by the interaction between the flexible wall of the tube and internal flow. We centre our attention on one-dimensional, time-dependent non-linear models. Classical works on this subject are, for example, [21], [28], and the many references therein. For more recent works see [2,7,10,12,13,29,31].

This paper is motivated by physical situations of medical interest in which certain properties that characterize compliant vessels, external pressures and body forces change rapidly, or even discontinuously. Physical quantities of interest are vessel wall thickness, equilibrium cross sectional area and Young's modulus. Prominent examples arise in the surgical treatment of Abdominal Aortic Aneurysms [35] that includes the insertion of stents. Stents are also implanted in veins [1] and in the ureter [6] in different circumstances. These devises do not always match the compliance properties of natural vessels and discontinuous jumps of physical properties may arise, influencing significantly the wave propagation phenomena associated with the fluid dynamics. External pressures and body forces are another source of potentially rapid or even discontinuous variations, which again will influence the wave phenomenon [21]. Here we formulate a mathematical model that allows for the discontinuous variations of certain vessel properties, all in the context of simplified one-dimensional flow. In spite of the very strong assumptions, we still expect the one-dimensional model to provide by itself useful information for practical purposes. Moreover, one-dimensional models are an integral part of large models in multiscale approaches [29] and thus the present work may also be useful in the construction of more realistic models.

In current models used for numerical simulation of blood flow phenomena the effect of the variation of the above mentioned quantities enters the equations in the form of source terms; see [31], for example. In particular, for external forces such as muscle forces, the corresponding source term involves a pressure gradient source term, analogous to the geometric source term given by bottom variation in shallow water models [32]. In the numerical analysis literature it is well known that such source terms are likely to cause serious numerical difficulties. An important issue is the construction *well balanced schemes* that achieve equilibrium between advective and source terms in the equations near the steady state [18, 23, 27]. The severity of the numerical difficulties increases as spatial gradients of the physical quantities of interest increase.

In this paper we formulate and study a simplified model in which discontinuities of three parameters are permitted, namely wall thickness, Young's modulus and cross sectional area at rest. Moreover we add two extra equations, one for the time variation of the external pressure p_e and one for the transport of a passive scalar. We study the mathematical properties of the resulting 6×6 hyperbolic system and obtain the exact solution of the associated Riemann problem. Exact solutions constitute reference solutions for assessing the performance of numerical methods intended for general use. Some preliminary results, obtained using a simpler model with one extra equation, have already been published [34]. Potentially, the proposed formulation would facilitate the numeri-