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A Two-Parameter Continuation Method for Rotating Two-Component Bose-Einstein Condensates in Optical Lattices

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Abstract. We study efficient spectral-collocation and continuation methods (SCCM) for rotating two-component Bose-Einstein condensates (BECs) and rotating two-component BECs in optical lattices, where the second kind Chebyshev polynomials are used as the basis functions for the trial function space. A novel two-parameter continuation algorithm is proposed for computing the ground state and first excited state solutions of the governing Gross-Pitaevskii equations (GPEs), where the classical tangent vector is split into two constraint conditions for the bordered linear systems. Numerical results on rotating two-component BECs and rotating two-component BECs in optical lattices are reported. The results on the former are consistent with the published numerical results.

AMS subject classifications: 65N35, 35P30, 35Q55

Key words: Spectral collocation method, second kind Chebyshev polynomials, periodic potential, bifurcation.

1 Introduction

Quantized vortex lattices in a rotating Bose-Einstein condensate (BEC) have been observed experimentally in the past decade [1–4]. Since then the study of quantized vortices plays a key role in superfluidity and superconductivity. While many interesting

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phenomena have been observed in single rotating component BEC [1–4], it is expected that a rich variety of static and dynamic phenomena will be found in rotating two component BECs. In this paper, we impose the effect of optical lattices on the physical system. At extremely low temperature, the mathematical model of rotating two-component BECs in optical lattices is described by the macroscopic wave functions $\Psi_j(\mathbf{x},t)$ $(j=1,2)$ whose evolution is governed by the coupled Gross-Pitaevskii equations (GPEs):

$$\mathbf{i}\frac{\partial}{\partial t}\Psi_j=-\frac{1}{2}\Delta\Psi_j+V(\mathbf{x})\Psi_j+P(\mathbf{x})\Psi_j+\left(\sum_{l=1}^{2}\eta_{jl}|\Psi_l|^2\right)\Psi_j-\omega L_z\Psi_j,\qquad \mathbf{x}\in\Omega\subset\mathbb{R}^2, \tag{1.1a}$$

$$\Psi_j(\mathbf{x},t)=0,\qquad t\geq 0,\quad \mathbf{x}\in\partial\Omega,\quad j=1,2, \tag{1.1b}$$

where $V(\mathbf{x})=(\gamma_x^2x^2+\gamma_y^2y^2)/2$ is the trapping potential with γ_x and γ_y the trap frequencies in the x- and y- direction, respectively, $P(\mathbf{x})=a_1\sin^2(\pi x/d_1)+a_2\sin^2(\pi y/d_2)$ is the periodic potential with a_1 and a_2 the recoil energies, and d_1 and d_2 are the distance between neighbor wells (lattice constants) in the x- and y- axis, respectively, $\Omega\subset\mathbb{R}^2$ is a bounded domain with piecewise smooth boundary $\partial\Omega$, ω is an angular velocity, and $L_z=xp_y-yp_x=-\mathbf{i}(x\partial y-y\partial x)$ is the z-component of the angular momentum $L=\mathbf{x}\times P$ with the momentum operator $P=-\mathbf{i}\nabla=(p_x,p_y,p_z)^T$. The intra-component interactions and inter-component interactions in the two-component BECs are represented by η_{jj} $(j=1,2)$ and η_{jl} $(j\neq l,\ j,l=1,2)$ respectively. An important invariant of the GPEs is the normalization of the wave functions

$$\int_\Omega|\Psi_j(\mathbf{x},t)|^2d\mathbf{x}=1,\qquad j=1,\,2,\quad t\geq 0. \tag{1.2}$$

Mueller and Ho [5] investigated the vortex lattice structure of (1.1) by assuming the lowest Landau level approximation and a perfect lattice. Kasamatsu *et al*. [6, 7] studied the vortex states structure of (1.1) with equal intra-component interaction strengths $\eta_{11}=\eta_{22}$ by varying the inter-component interaction constants η_{12} and η_{21} [7]. They also studied vortex states with and without internal Josephson coupling [8]. Recently, they studied the vortex sheet structure in rotating immiscible two-component BECs [9]. Zhang *et al*. studied the dynamics of (1.1) both analytically and numerically [10]. Recently, Wang studied numerical simulations for computing the ground state solutions of (1.1) [11].

Substituting the formula

$$\Psi_j(\mathbf{x},t)=e^{-\mathbf{i}\lambda_j t}\psi_j(\mathbf{x}),\qquad j=1,2,$$

into (1.1), we obtain a system of two stationary state nonlinear eigenvalue problems

$$\begin{aligned}\lambda_j\psi_j(\mathbf{x})=&-\frac{1}{2}\Delta\psi_j(\mathbf{x})+V(\mathbf{x})\psi_j(\mathbf{x})+P(\mathbf{x})\psi_j(\mathbf{x})+\left(\sum_{l=1}^{2}\eta_{jl}|\psi_l(\mathbf{x})|^2\right)\psi_j(\mathbf{x})\\&-\omega L_z\psi_j(\mathbf{x}),\qquad \mathbf{x}\in\Omega,\end{aligned} \tag{1.3a}$$

$$\psi_j(\mathbf{x})=0,\quad \text{on }\partial\Omega,\quad j=1,2, \tag{1.3b}$$