Commun. Comput. Phys.
doi: 10.4208/cicp.231209.150910s

Vol. **11**, No. 2, pp. 629-646
February 2012

Asymptotics for Helmholtz and Maxwell Solutions in 3-D Open Waveguides

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Received 23 December 2009; Accepted (in revised version) 15 September 2010

Available online 24 October 2011

Abstract. We extend classic Sommerfeld and Silver-Müller radiation conditions for bounded scatterers to acoustic and electromagnetic fields propagating over three isotropic homogeneous layers in three dimensions. If $\mathbf{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$, with x_3 denoting the direction orthogonal to the layers, standard conditions only hold for the outer layers in the region $|x_3|>\|\mathbf{x}\|^\gamma$, for $\gamma\in(1/4,1/2)$ and $\mathbf{x}$ large. For $|x_3|<\|\mathbf{x}\|^\gamma$ and inside the slab, asymptotic behavior depends on the presence of surface or guided modes given by the discrete spectrum of the associated operator.

AMS subject classifications: 35C15, 35Q60, 78A45

Key words: Open waveguide, radiation condition, Helmholtz equation.

1 Introduction

Existence and uniqueness of acoustic and electromagnetic (EM) waves over layered structures have for long remained unsolved problems. These so-called *open waveguides* possess solutions that are divided, according to the continuous and discrete parts of the operator spectrum, into radiative and guided modes, respectively [12]. Guided or surface modes decay differently at infinity than radiative modes, and consequently, standard radiation conditions do not suffice to guarantee uniqueness. In [4], this is overcome by introducing a modal condition on the volume of a 2-D rectangular waveguide with varying coefficients in the core. A different approach is presented in [3] wherein one of the outer layers is replaced by a Dirichlet condition and uniqueness is achieved via a generalized Fourier transform.

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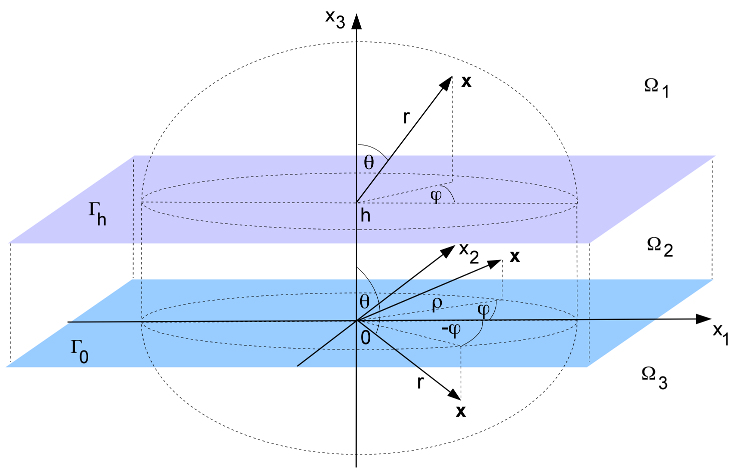

Figure 1: Coordinate and domain definitions.

In this work, we present rigorous asymptotics for outgoing acoustic and Maxwell waves in the time harmonic regime in $\mathbb{R}^3$ using the *limiting absorption principle* [9]. This constitutes a steppingstone towards a general existence result for open waveguides and uniqueness proofs in the fashion of [5]. On the application side, these precise asymptotic characterizations allow for the improvement of non-reflecting boundary conditions and perfectly matched layers (PML) based techniques in layered media [11].

1.1 Problem setting

1.1.1 Geometry and physical parameters

Let $h\in\mathbb{R}_+$, and define intervals $I_1:=(h,+\infty)$, $I_2:=(0,h)$ and $I_3:=(-\infty,0)$. We consider the following three-layer decomposition of $\mathbb{R}^3$ (see Fig. 1):

$$\Omega_1:=\{\mathbf{x}\in\mathbb{R}^3 : x_3\in I_1\},\qquad \Omega_2:=\{\mathbf{x}\in\mathbb{R}^3 : x_3\in I_2\},\qquad \Omega_3:=\{\mathbf{x}\in\mathbb{R}^3 : x_3\in I_3\},$$

with interfaces $\Gamma_0:=\overline{\Omega}_2\cap\overline{\Omega}_3$, $\Gamma_h:=\overline{\Omega}_2\cap\overline{\Omega}_1$, and $\Omega:=\bigcup_i\Omega_i$. Introduce hemispherical coordinates (r,θ,φ) with origin at $(0,0,h)$ for Ω_1 and at $(0,0,0)$ for Ω_3. That is, for $r\in\mathbb{R}_+$, $\varphi\in(0,2\pi)$, and either $\theta\in(0,\frac{\pi}{2})$ in Ω_1 or $\theta\in(\frac{\pi}{2},\pi)$ in Ω_3, we have the equivalences:

$$x_1=r\sin\theta\cos\varphi,\quad x_2=r\sin\theta\sin\varphi,\quad \text{and}\quad x_3=\begin{cases} h+\cos\theta, & x_3\in I_1,\\ \cos\theta, & x_3\in I_3.\end{cases}$$

In Ω_2, we employ cylindrical coordinates (ρ,φ,x_3) with $\rho>0$, $\varphi\in(0,2\pi)$ and $x_3\in I_2$, so that $x_1=\rho\cos\varphi$ and $x_2=\rho\sin\varphi$.

Each domain Ω_i, $i=1,2,3$, is characterized by different parameters according to the physical situation considered. For linear electromagnetism, relative permittivity and permeability coefficients, $\epsilon_i,\mu_i\in L^\infty(\Omega_i)$, are both real and positive. Inside Ω_i, the light speed