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An Efficient, Energy Stable Scheme for the Cahn-Hilliard-Brinkman System

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Abstract. We present an unconditionally energy stable and uniquely solvable finite difference scheme for the Cahn-Hilliard-Brinkman (CHB) system, which is comprised of a Cahn-Hilliard-type diffusion equation and a generalized Brinkman equation modeling fluid flow. The CHB system is a generalization of the Cahn-Hilliard-Stokes model and describes two phase very viscous flows in porous media. The scheme is based on a convex splitting of the discrete CH energy and is semi-implicit. The equations at the implicit time level are nonlinear, but we prove that they represent the gradient of a strictly convex functional and are therefore uniquely solvable, regardless of time step size. Owing to energy stability, we show that the scheme is stable in the time and space discrete $\ell^\infty(0,T;H_h^1)$ and $\ell^2(0,T;H_h^2)$ norms. We also present an efficient, practical nonlinear multigrid method – comprised of a standard FAS method for the Cahn-Hilliard part, and a method based on the Vanka smoothing strategy for the Brinkman part – for solving these equations. In particular, we provide evidence that the solver has nearly optimal complexity in typical situations. The solver is applied to simulate spinodal decomposition of a viscous fluid in a porous medium, as well as to the more general problems of buoyancy- and boundary-driven flows.

AMS subject classifications: 65M06, 65M12, 65M55, 76T99

Key words: Cahn-Hilliard equation, Stokes equations, Brinkman equation, finite difference methods, nonlinear multigrid, convex splitting, energy stability.

1 Introduction

1.1 Problem definition and background

Consider the Ginzburg-Landau energy of a binary fluid with constant, uniform mass density [5]

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$$E(\phi)=\int_\Omega \left\{ \frac{1}{4\epsilon}\left(\phi^2-1\right)^2+\frac{\epsilon}{2}|\nabla\phi|^2 \right\} d\mathbf{x}, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^D$, $D=2$ or 3, $\phi:\Omega\to\mathbb{R}$ is the concentration field, and ϵ is a positive constant. The phase equilibria are represented by the pure fluids $\phi=\pm1$. The dynamical Cahn-Hilliard-Brinkman equations we consider are

$$\partial_t\phi=\epsilon\nabla\cdot(M(\phi)\nabla\mu)-\nabla\cdot(\phi\mathbf{u}), \tag{1.2}$$

$$-\nabla\cdot[\nu(\phi)\mathsf{D}(\mathbf{u})]+\eta(\phi)\mathbf{u}=-\nabla p-\gamma\phi\nabla\mu, \tag{1.3}$$

$$\nabla\cdot\mathbf{u}=0, \tag{1.4}$$

where $M(\phi)$ is a mobility that incorporates the Peclet number; μ is the chemical potential

$$\mu:=\delta_\phi E=\frac{1}{\epsilon}(\phi^3-\phi)-\epsilon\Delta\phi; \tag{1.5}$$

$\gamma>0$ is a surface tension parameter; $\mathbf{u}$ is the fluid velocity; p is a pressure; $\nu(\cdot)>0$ is the fluid viscosity; $\eta(\cdot)>0$ is the fluid permeability; and $\mathsf{D}(\mathbf{u})=\nabla\mathbf{u}+(\nabla\mathbf{u})^T$.

We assume that M, ν, $\eta\in C^\infty$, and $M(x)\geq M_0>0$, $\eta(x)\geq\eta_0>0$, and $\nu(x)\geq\nu_0>0$, for all $x\in\mathbb{R}$. For example, we shall frequently use a regularized degenerate mobility of the form

$$M(\phi)=\frac{1}{\mathsf{Pe}}\sqrt{(1+\phi)^2(1-\phi)^2+\epsilon^2}\geq\frac{\epsilon}{\mathsf{Pe}}>0, \tag{1.6}$$

where $\mathsf{Pe}>0$ is the Peclet number, which might be dependent upon ϵ. We assume that the system (1.2)-(1.4) is supplemented with the boundary conditions $\mathbf{u}|_{\partial\Omega}=0$, and $\partial_n\phi|_{\partial\Omega}=\partial_n\mu|_{\partial\Omega}=0$. The latter conditions represent local thermodynamic equilibrium on the boundary. With this set of boundary conditions, the system (1.2)-(1.4) is mass conservative and energy dissipative, and the dissipation rate is readily found to be [8,14,15,17]

$$d_tE=-\epsilon(M(\phi)\nabla\mu,\nabla\mu)_{L^2}-\frac{1}{\gamma}(\eta(\phi)\mathbf{u},\mathbf{u})_{L^2}-\frac{1}{2\gamma}(\nu(\phi)\mathsf{D}(\mathbf{u}),\mathsf{D}(\mathbf{u}))_{L^2}\leq0. \tag{1.7}$$

Eq. (1.3) is a generalized Brinkman equation [2] that incorporates a diffuse interface surface tension force. The Cahn-Hilliard-Brinkman (CHB) system (1.2)-(1.4) was recently proposed as a model for phase separation and coarsening of a binary fluid in a Brinkman porous medium [18]. The authors showed the existence of a logarithmically slow coarsening regime that arises when the phase domains are comparable to the average pore size. The system (1.2)-(1.4) is also closely related to models of tumor growth [20,33,34] which include an additional mass source for volumetric growth. When the surface tension vanishes, *i.e.*, $\gamma=0$, the CHB system reduces to the Cahn-Hilliard equation [3]. A generalized Stokes equation is obtained when $\eta\equiv0$ in (1.3). The system (1.2)-(1.4) is a simplified version of the model derived by Lee *et al.* [14,15], which they used to describe gravity-driven, density-mismatched, two-phase flow. We remark that the CHB system