An Efficient, Energy Stable Scheme for the Cahn-Hilliard-Brinkman System

Craig Collins\textsuperscript{1}, Jie Shen\textsuperscript{2} and Steven M. Wise\textsuperscript{1,*}

\textsuperscript{1} Department of Mathematics, University of Tennessee, Knoxville, TN 37912, USA.
\textsuperscript{2} Department of Mathematics, Purdue University, West Lafayette, IN 47907, USA.

Received 17 December 2011; Accepted (in revised version) 13 April 2012

Communicated by Bo Li
Available online 21 September 2012

Abstract. We present an unconditionally energy stable and uniquely solvable finite difference scheme for the Cahn-Hilliard-Brinkman (CHB) system, which is comprised of a Cahn-Hilliard-type diffusion equation and a generalized Brinkman equation modeling fluid flow. The CHB system is a generalization of the Cahn-Hilliard-Stokes model and describes two phase very viscous flows in porous media. The scheme is based on a convex splitting of the discrete CH energy and is semi-implicit. The equations at the implicit time level are nonlinear, but we prove that they represent the gradient of a strictly convex functional and are therefore uniquely solvable, regardless of time step size. Owing to energy stability, we show that the scheme is stable in the time and space discrete $\ell^\infty(0,T; H^1_0)$ and $\ell^2(0,T; H^2_0)$ norms. We also present an efficient, practical nonlinear multigrid method – comprised of a standard FAS method for the Cahn-Hilliard part, and a method based on the Vanka smoothing strategy for the Brinkman part – for solving these equations. In particular, we provide evidence that the solver has nearly optimal complexity in typical situations. The solver is applied to simulate spinodal decomposition of a viscous fluid in a porous medium, as well as to the more general problems of buoyancy- and boundary-driven flows.

AMS subject classifications: 65M06, 65M12, 65M55, 76T99

Key words: Cahn-Hilliard equation, Stokes equations, Brinkman equation, finite difference methods, nonlinear multigrid, convex splitting, energy stability.

1 Introduction

1.1 Problem definition and background

Consider the Ginzburg-Landau energy of a binary fluid with constant, uniform mass density [5]

\textsuperscript{*}Corresponding author. Email addresses: craig.collins@math.utk.edu (C. Collins), shen@math.purdue.edu (J. Shen), swise@math.utk.edu (S. M. Wise)
\[ E(\phi) = \int_{\Omega} \left\{ \frac{1}{4} (\phi^2 - 1)^2 + \frac{\epsilon}{2} |\nabla \phi|^2 \right\} \, dx, \]  

(1.1)

where \( \Omega \subset \mathbb{R}^D, D = 2 \) or 3, \( \phi : \Omega \to \mathbb{R} \) is the concentration field, and \( \epsilon \) is a positive constant. The phase equilibria are represented by the pure fluids \( \phi = \pm 1 \). The dynamical Cahn-Hilliard-Brinkman equations we consider are

\[
\begin{align*}
\partial_t \phi &= \epsilon \nabla \cdot (M(\phi) \nabla \mu) - \nabla \cdot (\phi \mathbf{u}), \\
-\nabla \cdot [\nu(\phi) D(\mathbf{u})] + \eta(\phi) \mathbf{u} &= -\nabla p - \gamma \phi \nabla \mu, \\
\nabla \cdot \mathbf{u} &= 0,
\end{align*}
\]

(1.2) - (1.4)

where \( M(\phi) \) is a mobility that incorporates the Peclet number; \( \mu \) is the chemical potential

\[ \mu := \delta \phi E = \frac{1}{\epsilon} (\phi^3 - \phi) - \epsilon \Delta \phi; \]

(1.5)

\( \gamma > 0 \) is a surface tension parameter; \( \mathbf{u} \) is the fluid velocity; \( p \) is a pressure; \( \nu(\cdot) > 0 \) is the fluid viscosity; \( \eta(\cdot) > 0 \) is the fluid permeability; and \( D(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^T \).

We assume that \( M, \nu, \eta < \mathcal{C}^\infty \), and \( M(x) \geq M_0 > 0, \eta(x) \geq \eta_0 > 0, \) and \( \nu(x) \geq \nu_0 > 0 \), for all \( x \in \mathbb{R} \). For example, we shall frequently use a regularized degenerate mobility of the form

\[ M(\phi) = \frac{1}{\text{Pe}} \sqrt{\frac{(1+\phi)^2(1-\phi)^2 + \epsilon^2}{\text{Pe}}} > 0, \]

(1.6)

where \( \text{Pe} > 0 \) is the Peclet number, which might be dependent upon \( \epsilon \). We assume that the system (1.2)-(1.4) is supplemented with the boundary conditions \( \mathbf{u} |_{\partial \Omega} = 0 \), and \( \partial_n \phi |_{\partial \Omega} = \partial_n \mu |_{\partial \Omega} = 0 \). The latter conditions represent local thermodynamic equilibrium on the boundary. With this set of boundary conditions, the system (1.2)-(1.4) is mass conservative and energy dissipative, and the dissipation rate is readily found to be \([8, 14, 15, 17]\)

\[ d_t E = -\epsilon (M(\phi) \nabla \mu, \nabla \mu)_{L^2} - \frac{1}{\gamma} (\eta(\phi) \mathbf{u}, \mathbf{u})_{L^2} - \frac{1}{2\gamma} (\nu(\phi) D(\mathbf{u}), D(\mathbf{u}))_{L^2} \leq 0. \]

(1.7)

Eq. (1.3) is a generalized Brinkman equation \([2]\) that incorporates a diffuse interface surface tension force. The Cahn-Hilliard-Brinkman (CHB) system (1.2)-(1.4) was recently proposed as a model for phase separation and coarsening of a binary fluid in a Brinkman porous medium \([18]\). The authors showed the existence of a logarithmically slow coarsening regime that arises when the phase domains are comparable to the average pore size. The system (1.2)-(1.4) is also closely related to models of tumor growth \([20, 33, 34]\) which include an additional mass source for volumetric growth. When the surface tension vanishes, i.e., \( \gamma = 0 \), the CHB system reduces to the Cahn-Hilliard equation \([3]\). A generalized Stokes equation is obtained when \( \eta \equiv 0 \) in (1.3). The system (1.2)-(1.4) is a simplified version of the model derived by Lee et al. \([14, 15]\), which they used to describe gravity-driven, density-mismatched, two-phase flow. We remark that the CHB system