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A Discussion on Two Stochastic Elliptic Modeling Strategies

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Abstract. Based on the study of two commonly used stochastic elliptic models: I: $-\nabla\cdot(a(\boldsymbol{x},\omega)\cdot\nabla u(\boldsymbol{x},\omega))=f(\boldsymbol{x})$ and II: $-\nabla\cdot(a(\boldsymbol{x},\omega)\diamond\nabla u(\boldsymbol{x},\omega))=f(\boldsymbol{x})$, we constructed a new stochastic elliptic model III: $-\nabla\cdot\big((a^{-1})^{\diamond(-1)}\diamond\nabla u(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x})$, in [20]. The difference between models I and II is twofold: a scaling factor induced by the way of applying the Wick product and the regularization induced by the Wick product itself. In [20], we showed that model III has the same scaling factor as model I. In this paper we present a detailed discussion about the difference between models I and III with respect to the two characteristic parameters of the random coefficient, i.e., the standard deviation σ and the correlation length l_c. Numerical results are presented for both one- and two-dimensional cases.

AMS subject classifications: 60H15, 65C20, 65C30

Key words: Wiener chaos, stochastic PDE, stochastic modeling, stochastic finite element method.

1 Introduction

Stochastic elliptic models are of fundamental importance for the stochastic modeling of physical and engineering applications [9, 15]. The two commonly studied stochastic elliptic models in literature include

$$\text{Model I:}\qquad -\nabla\cdot\big(a(\boldsymbol{x},\omega)\nabla u_I(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x}), \tag{1.1a}$$

$$\text{Model II:}\qquad -\nabla\cdot\big(a(\boldsymbol{x},\omega)\diamond\nabla u_{II}(\boldsymbol{x},\omega)\big)=f(\boldsymbol{x}), \tag{1.1b}$$

where $\boldsymbol{x}\in\mathbb{R}^d$, $d=1,2,3$, ω indicates randomness, $a(\boldsymbol{x},\omega)$ a non-negative random process and $\diamond$ the Wick product. Based on the properties of $a(\boldsymbol{x},\omega)$, models I and II can be adapted

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for different applications. For example, if the random coefficient $a(\boldsymbol{x},\omega)$ is ergodic and has two wildly separated scales, model I becomes a typical stochastic multi-scale elliptic model. In this work, we consider a general case, where we assume that $a(\boldsymbol{x},\omega)$ is log-normal and the underlying Gaussian random process is homogeneous stationary and ergodicity is not required. For such a set-up, we refer to [1, 2, 6–8, 14] and references therein for theoretical and numerical studies for model I and [9–11, 17–19] and references therein for model II.

The difference between models I and II is twofold: a scaling factor induced by the way of applying the Wick product and the regularization induced by the Wick product itself. It was shown in [20] that the scaling factor is an exponential function of the variance of the underlying Gaussian random process of $a(\boldsymbol{x},\omega)$. By applying the Wick product in a different way, a new stochastic elliptic model

$$\text{Model III:}\quad -\nabla\cdot\left((a^{-1})^{\diamond(-1)}\diamond\nabla u_{III}(\boldsymbol{x},\omega)\right)=f(\boldsymbol{x}) \tag{1.2}$$

was proposed in [20], whose solution has the same scaling factor as model I. Numerical experiments showed that for one-dimensional problems the solutions of models I and III can be very close to each other, which implies that the regularization effect induced by the Wick product is relatively small.

In this work, we continue the study on the two stochastic modeling strategies based on the regular product and the Wick product. We will focus on the regularization effect induced by the Wick product by examining the difference between models I and III with respect to the standard deviation σ and the correlation length l_c of the underlying Gaussian process of a log-normal random coefficient $a(\boldsymbol{x},\omega)$. Asymptotic analysis shows that the difference between the solutions of models I and III is of second order with respect to σ, i.e.,

$$\|u_I-u_{III}\|\sim C(l_c)\sigma^2.$$

Such a fact is independent of the physical dimension d. Thus model III can provide most of the information given by model I when σ is relatively small. In particular, when l_c goes to infinity, the constant $C(l_c)$ will decay to zero. It is shown that the solutions of models I and III converge to each other as l_c goes to zero, which is a fact that is only true for one-dimensional problems. Analysis and numerical results also show that the solutions of models I and III are almost linear with respect to each other in a statistical sense if σ is relatively small.

This paper is organized as follows: in Section 2 we introduce the weighted Wiener chaos space, which is a uniform theoretical framework for models I-III. A detailed description of the three stochastic elliptic models is given in Section 3. We present some theoretical studies about the difference between models I and III in Section 4. Numerical results for two-dimensional problems are given in Section 5 followed by a summary section.