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Truncation Errors, Exact and Heuristic Stability Analysis of Two-Relaxation-Times Lattice Boltzmann Schemes for Anisotropic Advection-Diffusion Equation

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Abstract. This paper establishes relations between the stability and the high-order truncated corrections for modeling of the mass conservation equation with the two-relaxation-times (TRT) collision operator. First we propose a simple method to derive the truncation errors from the exact, central-difference type, recurrence equations of the TRT scheme. They also supply its equivalent three-time-level discretization form. Two different relationships of the two relaxation rates nullify the third (advection) and fourth (pure diffusion) truncation errors, for any linear equilibrium and any velocity set. However, the two relaxation times alone cannot remove the leading-order advection-diffusion error, because of the intrinsic fourth-order numerical diffusion. The truncation analysis is carefully verified for the evolution of concentration waves with the anisotropic diffusion tensors. The anisotropic equilibrium functions are presented in a simple but general form, suitable for the minimal velocity sets and the d2Q9, d3Q13, d3Q15 and d3Q19 velocity sets. All anisotropic schemes are complemented by their exact necessary von Neumann stability conditions and equivalent finite-difference stencils. The sufficient stability conditions are proposed for the most stable (OTRT) family, which enables modeling at any Peclet numbers with the same velocity amplitude. The heuristic stability analysis of the fourth-order truncated corrections extends the optimal stability to larger relationships of the two relaxation rates, in agreement with the exact (one-dimensional) and numerical (multi-dimensional) stability analysis. A special attention is put on the choice of the equilibrium weights. By combining accuracy and stability predictions, several strategies for selecting the relaxation and free-tunable equilibrium parameters are suggested and applied to the evolution of the Gaussian hill.

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Key words: Two-relaxation-times Lattice Boltzmann scheme, AADE, truncation errors, von Neumann stability analysis, numerical diffusion, heuristic stability analysis.

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1 Introduction

The two-relaxation-times (TRT) Lattice Boltzmann model is suitable for solving hydrodynamic equations [16–19], and the anisotropic, linear or non-linear, advection-diffusion equations (AADE), [12–15, 44–46]. This paper further investigates the role of the free-tunable relaxation and equilibrium parameters, with the objective to find the best balance between error minimization and robustness. So far, this work extends the mathematical analysis [7, 8, 36] of the multiple-relaxation-times (MRT) operators, TRT operator [12,17,20,34,44], single-relaxation-time BGK operator [26,43] and high-order equilibrium BGK schemes [6,49]. In fact, the TRT model combines the simplicity and efficiency of the BGK operator [41] with a specific capability of the multiple-relaxation-times operators [4,22,23,27–29] to control their numerical solutions with the help of the "free" relaxation parameters [1,11,29,36,37]. The TRT model has only one free eigenvalue function, say Λ^- for the anti-symmetric modes modeling the Navier-Stokes equations, and Λ^+ for the symmetric modes modeling the AADE. The viscous and diffusion coefficients are then defined by Λ^+ and Λ^-, respectively. The rigorous analysis of the exact form of the steady state conservation equations undoubtedly shows that the LBE schemes need the distinguished relaxation rates for the different parity eigenmodes, to avoid a non-linear dependence of the truncation spatial errors on the selected transport coefficients [30]. The TRT operator is the minimal scheme which allows a full control of the *steady state* hydrodynamic and advection-diffusion solutions by the non-dimensional physical parameters, as Reynolds and Peclet numbers, provided that the free product $\Lambda=\Lambda^+\Lambda^-$ is fixed. The BGK subclass of the TRT scheme lacks this feature since $\Lambda^-=\Lambda^+$. However, in addition to non-dimensional governing numbers, the *steady state* TRT solutions are controlled by Λ. For example, the permeability of a porous media calculated with a fixed Λ is independent of the viscosity coefficient and depend only on the assigned value of Λ, [5, 11, 18]. Even if the observed variation of the solutions in reasonable Λ-interval is often comparable with the experimental incertitude [33], the question how to select Λ properly has not only methodological but quite practical interest.

Several specific values could be listed. Two functions,

$$\Lambda=3\delta^2/4 \quad \text{and} \quad \Lambda=3\delta^2/2,$$

enable the exact location of solid walls with second-order accurate boundary schemes, for plane and diagonal Poiseuille flow, respectively, when the distance to boundary is equal to δ, with $\delta=\frac{1}{2}$ for the bounce-back rule (see [19] and references herein). Similar solutions can be derived for the pressure or AADE Dirichlet boundary schemes, [9, 13, 18]. The MRT schemes often apply the TRT solutions to relevant eigenvalue relationships, [2, 5, 11, 40, 47]. In bulk, the two particular TRT configurations, $\Lambda=\frac{1}{12}$ for the third-order and $\Lambda=\frac{1}{6}$ for the fourth order, nullify the corresponding coefficients of the *steady state* Chapman-Enskog expansion [17], improving the accuracy and grid convergence of the scheme for any equilibrium. However, the *exact time-dependent recurrence* equations of