Commun. Comput. Phys.
doi: 10.4208/cicp.101010.250511a

Vol. **11**, No. 3, pp. 951-984
March 2012

Composite Coherent States Approximation for One-Dimensional Multi-Phased Wave Functions

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Received 10 October 2010; Accepted (in revised version) 25 May 2011

Available online 18 November 2011

Abstract. The coherent states approximation for one-dimensional multi-phased wave functions is considered in this paper. The wave functions are assumed to oscillate on a characteristic wave length $\mathcal{O}(\epsilon)$ with $\epsilon \ll 1$. A parameter recovery algorithm is first developed for single-phased wave function based on a moment asymptotic analysis. This algorithm is then extended to multi-phased wave functions. If cross points or caustics exist, the coherent states approximation algorithm based on the parameter recovery will fail in some local regions. In this case, we resort to the windowed Fourier transform technique, and propose a composite coherent states approximation method. Numerical experiments show that the number of coherent states derived by the proposed method is much less than that by the direct windowed Fourier transform technique.

AMS subject classifications: 33F05, 68W25

Key words: Coherent state, high frequency, windowed Fourier transform.

1 Introduction

This paper aims at an efficient coherent states approximation method for wave functions oscillating at most on the $\mathcal{O}(\epsilon)$ scale with $\epsilon \ll 1$. By *coherent state* we mean a function of the following form

$$\tilde{A}\exp\left(\frac{i}{\epsilon}\left(p(x-q)+\frac{\gamma}{2}(x-q)^2\right)\right), \tag{1.1}$$

where $q\in\mathbb{R}$ is termed *spatial center*, $p\in\mathbb{R}$ *momentum*, $\gamma\in\mathbb{C}$ *spread*, and $\tilde{A}\in\mathbb{C}$ *amplitude*. The imaginary part of γ should be positive, which renders to the coherent state function a Gaussian profile centered at point q. A *coherent states approximation* (CSA) is a set of

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coherent states parameterized by $\{q_j,p_j,\gamma_j,\tilde{A}_j\}$, whose summation approximates a given family of wave functions $u_\epsilon(x)$ by an asymptotic error $\mathcal{O}(\mathscr{E})$ with $\lim_{\epsilon\to 0}\mathscr{E}=0$, i.e.,

$$u_\epsilon(x)=\sum_j \tilde{A}_j \exp\left(\frac{i}{\epsilon}\left(p_j(x-q_j)+\frac{\gamma_j}{2}(x-q_j)^2\right)\right)+\mathcal{O}(\mathscr{E}). \tag{1.2}$$

The CSA problem exists in various disciplines, such as quantum chemistry [6–8, 10], geophysics [4, 5, 9, 14], and signal processing [2, 11, 13]. For example, in geophysics, if one wants to perform seismic migration with the Gaussian beam approach, the first issue faced by the practitioner is to decompose the acquired seismic signal into a set of coherent state functions. The approximating accuracy is very important for an accurate and reliable exploration. On the other hand, however, the number of derived coherent states should not be too large so that the migration algorithm can be implemented within the limited computing power. This dilemma situation also appears in quantum mechanics whenever a semi-classical approximation for the propagator based on Gaussian coherent states is employed to evolve the quantum wave field.

It is well known that the following set of coherent states

$$\varphi_{pq}(x)=\frac{1}{(\pi\epsilon a)^{1/4}(2\pi\epsilon)^{\frac{1}{2}}}\exp\left(\frac{i}{\epsilon}\left(p(x-q)+\frac{i}{2a}(x-q)^2\right)\right),$$

parameterized in the phase space $\mathbb{R}_p\times\mathbb{R}_q$, form a tight frame in $L^2(\mathbb{R})$ for any $a>0$ (see Appendix C). This means that for any $f(x)\in L^2(\mathbb{R})$ the following holds

$$f(x)=\iint_{\mathbb{R}_p\times\mathbb{R}_q}(f,\varphi_{pq})\varphi_{pq}(x)dpdq. \tag{1.3}$$

Here $(\cdot,\cdot)$ indicates the standard $L^2(\mathbb{R})$ inner product. Since

$$(f,f)=\iint_{\mathbb{R}_p\times\mathbb{R}_q}|(f,\varphi_{pq})|^2dpdq,$$

the "coordinates" (f,φ_{pq}) can be taken as the energy spectra of f in the phase space. A discrete CSA is thus derived by applying a suitable numerical quadrature on (1.3). For example, using the trapezoidal rule gives

$$f(x)\approx\Delta p\Delta q\sum_{j,k\in\mathbb{Z}}(f,\varphi_{p_jq_k})\varphi_{p_jq_k}(x), \tag{1.4}$$

where $p_j=j\Delta p$ and $q_k=k\Delta q$, with Δp and Δq being the momentum and spatial stepsizes respectively.

There exists an alternative way based on the dual frame technique to arrive at a CSA as (1.4) [2, 16]. Let a and b be two positive constants on the scale of $\mathcal{O}(1)$. Set $q_j=j\sqrt{\epsilon a}/b$ and

$$h_j(x)=h_j(a,b,x)=\frac{1}{\sqrt{2\pi b^2}}\exp\left(-\frac{(x-q_j)^2}{2\epsilon a}\right). \tag{1.5}$$