An Adaptive Time-Stepping Strategy for the Cahn-Hilliard Equation

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Abstract. This paper studies the numerical simulations for the Cahn-Hilliard equation which describes a phase separation phenomenon. The numerical simulation of the Cahn-Hilliard model needs very long time to reach the steady state, and therefore large time-stepping methods become useful. The main objective of this work is to construct the unconditionally energy stable finite difference scheme so that the large time steps can be used in the numerical simulations. The equation is discretized by the central difference scheme in space and fully implicit second-order scheme in time. The proposed scheme is proved to be unconditionally energy stable and mass-conservative. An error estimate for the numerical solution is also obtained with second order in both space and time. By using this energy stable scheme, an adaptive time-stepping strategy is proposed, which selects time steps adaptively based on the variation of the free energy against time. The numerical experiments are presented to demonstrate the effectiveness of the adaptive time-stepping approach.

1 Introduction

The Cahn-Hilliard equation arises as a phenomenological continuum model for the two mixture components and was originally introduced in [1]:

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$$\frac{\partial u}{\partial t}+\Delta(u-u^3+\kappa\Delta u)=0, \qquad (\boldsymbol{x},t)\in\Omega\times\mathrm{R}^+, \tag{1.1a}$$

$$u(\boldsymbol{x},0)=u_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega, \tag{1.1b}$$

$$u(\cdot,t) \text{ is subject to periodic boundary condition}, \tag{1.1c}$$

where the domain $\Omega=(0,L_1)\times(0,L_2)$ is an open set in R^2, κ is a positive constant and $u_0(\boldsymbol{x})$ is a given function.

There have been an enormous amount of research work, and still growing, on the mathematical and numerical analysis of the Cahn-Hilliard equation. In [2, 14] the steady state solutions of the Cahn-Hilliard equation were studied. Elliott and Zheng proved the existence and uniqueness of the solution for the Cahn-Hilliard equation in [6]. It is difficult to find the analytical solutions, so many efforts have been made on the numerical simulations for the Cahn-Hilliard equation recently. Finite element methods have been first studied on solving the Cahn-Hilliard equation by Elliott et al. in [7–9]. In [5], Du and Nicolaides proposed a fully finite element method and proved the convergence without assumptions beyond those necessary for existence and uniqueness of the differential equation. In [17], a linearized finite difference scheme was derived using the method of reduction of order. Solvability and convergence were studied, but the mass conservation is not preserved and only conditional stability was obtained. In [4], a conservative nonlinear finite difference scheme was proposed, which is unconditionally stable in L^∞-norm and conserves the total mass. However, the energy-based stability was not discussed. In [10], a conservative difference method was proposed for solving the one-dimensional Cahn-Hilliard equation and it was proved to be unconditionally stable in the sense of energy decay. Most recently, an unconditionally stable finite difference scheme was developed to solve the Cahn-Hilliard-Hele-Shaw system of equations in [19].

Since the simulation of Cahn-Hilliard model needs very long time to reach the steady state, the large time-stepping method is needed. In [11], a large time-stepping method was proposed for simulating the Cahn-Hilliard equation. The time step can be increased by adding a linear term to the classical semi-implicit finite difference scheme. However, the adding term is dependent on the unknown numerical solution. In [20], the same large time-stepping method was applied to the epitaxial growth models. In [21], a nonconforming finite element method coupled with the convexity-splitting scheme for the discretization in temporal variable was proposed to solve the Cahn-Hilliard equation. An artificial extra term was also introduced to adjust the dissipation of the energy, which was dependent on the unknown numerical solution.

In this paper, we present a finite difference scheme for solving the two-dimensional Cahn-Hilliard equation, which discretizes the equation by the central difference scheme in space and fully implicit second order scheme in time. The stability and error estimate will be analyzed. It will be proved that this finite difference scheme is unconditionally energy stable and mass-conservative, which guarantees that the large time steps can be used in the numerical simulations of the Cahn-Hilliard equation. However, constant