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A High Frequency Boundary Element Method for Scattering by Convex Polygons with Impedance Boundary Conditions

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Abstract. We consider scattering of a time harmonic incident plane wave by a convex polygon with piecewise constant impedance boundary conditions. Standard finite or boundary element methods require the number of degrees of freedom to grow at least linearly with respect to the frequency of the incident wave in order to maintain accuracy. Extending earlier work by Chandler-Wilde and Langdon for the sound soft problem, we propose a novel Galerkin boundary element method, with the approximation space consisting of the products of plane waves with piecewise polynomials supported on a graded mesh with smaller elements closer to the corners of the polygon. Theoretical analysis and numerical results suggest that the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency of the incident wave.

AMS subject classifications: 35J05, 65N38, 65R20

Key words: Boundary integral equation method, high frequency scattering, convex polygons, impedance boundary conditions.

1 Introduction

In this paper we consider two-dimensional scattering of a time-harmonic incident plane wave $u^i(\mathbf{x}) = \mathrm{e}^{ik\mathbf{x}\cdot\mathbf{d}}$, $\mathbf{x}\in\mathbb{R}^2$, where the unit vector $\mathbf{d}$ is the direction of propagation and $k>0$ is the wavenumber of the incident wave, by a bounded Lipschitz domain $\Omega\subset\mathbb{R}^2$, with impedance boundary conditions holding on $\Gamma:=\partial\Omega$. Define $D:=\mathbb{R}^2\setminus\overline{\Omega}$ to be the unbounded domain exterior to Ω, let $\gamma^+: H^1(D)\to H^{1/2}(\Gamma)$ and $\gamma^-: H^1(\Omega)\to H^{1/2}(\Gamma)$ denote the exterior and interior trace operators, respectively, and, where $H^1(G,\Delta):=\{v\in$

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$H^1(G):\Delta v\in L^2(G)\}$, let $\partial_{\mathbf{n}}^+ : H^1(D,\Delta)\to H^{-1/2}(\Gamma)$ and $\partial_{\mathbf{n}}^- : H^1(\Omega,\Delta)\to H^{-1/2}(\Gamma)$ denote the exterior and interior normal derivative operators, respectively. (All of $\gamma^\pm$ and $\partial_{\mathbf{n}}^\pm$ are well-defined as bounded linear operators, see [15], where also our various function space notations are defined.) Then the scattering problem we consider is: given $\beta\in L^\infty(\Gamma)$, find the total field $u^t\in C^2(D)\cap H^1_{\rm loc}(D)$ such that

$$\Delta u^t + k^2 u^t = 0, \qquad \text{in } D, \tag{1.1}$$

$$\partial_{\mathbf{n}}^+ u^t + \mathrm{i}k\beta\gamma^+ u^t = 0, \quad \text{on } \Gamma, \tag{1.2}$$

and such that the scattered field $u^s := u^t - u^i$ satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r}(\mathbf{x}) - \mathrm{i}k u^s(\mathbf{x}) = o\left(r^{-1/2}\right), \tag{1.3}$$

as $r := |\mathbf{x}| \to \infty$, uniformly with respect to $\mathbf{x}/|\mathbf{x}|$.

It is a standard result that this boundary value problem is uniquely solvable if $\mathrm{Re}\beta \geq 0$, which physically is a condition that the impedance boundary does not emit energy. (See [9] for a proof in the case that Γ is C^2, and [15, Lemma 9.9, Exercise 9.5] for the main ideas for the extension to the case of Lipschitz Γ.) Our concern in this paper is to develop a novel and very effective high frequency boundary element method for the particular case when Ω is a convex polygon and β is constant on each side of Γ, corresponding to an obstacle made up from several, homogeneous, materials, each with a different relative surface admittance.

The problem (1.1)–(1.3) has received significant recent attention in the literature [1, 19, 22]. Standard boundary or finite element approximations suffer from the requirement that the number of degrees of freedom must increase at least linearly with respect to k in order to maintain accuracy. Although asymptotic schemes can provide reasonable approximations when k is very large [12], there thus exists a wide range of frequencies for which numerical schemes are prohibitively expensive whilst asymptotic approaches are insufficiently accurate. This difficulty for scattering problems has been well documented in the literature in recent years, and numerous novel approaches to reducing the computational cost for moderate to large k have been proposed. The scattering problem that has received the most attention in the literature is the sound soft problem, i.e. (1.1) and (1.3) with the boundary condition $\gamma^+ u^t = 0$ on Γ replacing (1.2). Using a boundary element approach, with a hybrid approximation space consisting of the product of plane waves with piecewise polynomials, very efficient schemes have been developed for scattering by smooth obstacles [10] and by convex polygons [6], with in each case the number of degrees of freedom required to achieve a prescribed level of accuracy depending only very mildly on k. In contrast, the only numerical scheme for (1.1)–(1.3) that we are aware of that has been developed specifically for the purpose of efficiency at high frequencies is that in [19], where a circle of piecewise constant impedance is considered. There the approximation space is enriched with plane waves traveling in multiple directions; this reduces the number of degrees of freedom required per wavelength from ten to three (for