Commun. Comput. Phys.
doi: 10.4208/cicp.120210.240511a

Vol. **11**, No. 5, pp. 1643-1672
May 2012

A Well-Posed and Discretely Stable Perfectly Matched Layer for Elastic Wave Equations in Second Order Formulation

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Received 12 February 2010; Accepted (in revised version) 24 May 2011

Communicated by Jan S. Hesthaven

Available online 12 January 2012

Abstract. We present a well-posed and discretely stable perfectly matched layer for the anisotropic (and isotropic) elastic wave equations without first re-writing the governing equations as a first order system. The new model is derived by the complex coordinate stretching technique. Using standard perturbation methods we show that complex frequency shift together with a chosen real scaling factor ensures the decay of eigen-modes for all relevant frequencies. To buttress the stability properties and the robustness of the proposed model, numerical experiments are presented for anisotropic elastic wave equations. The model is approximated with a stable node-centered finite difference scheme that is second order accurate both in time and space.

AMS subject classifications: 35B35, 35L05, 35L15, 37C75

Key words: Perfectly matched layer, well-posedness, stability, hyperbolicity, elastic waves.

1 Introduction

Perfectly matched layers (PML) have since the introduction [3], emerged as a standard non-reflecting boundary closure for many wave propagation problems. The basic properties of a PML can be found in [6]. In this paper we consider linear, anisotropic elastodynamics in two space dimensions. Equations describing the dynamics are usually derived via Newton's law, which connects acceleration and force, and yields a second order system (in both time and space) for the displacements. The system is hyperbolic, and by introducing suitable variables the model can be rewritten as a hyperbolic first

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order system. PMLs for elasto-dynamics are usually derived from the first order formulation [4, 5, 23]. This is also the case for other hyperbolic systems that naturally come in second order formulation, like the standard wave equation.

However, there are several advantages with using the second order formulation. The first order formulation requires more variables, and it introduces a new wave with zero wave speed. Also, in many cases a straightforward discretization of the first order formulation introduces high frequency spurious modes. In this paper we construct a PML for the second order equations of linear, anisotropic elasto-dynamics in two space dimensions without first rewriting the equations as a first order system. By construction the PML is perfectly matched, but there is no guarantee that all solutions decay with time. The analysis of temporal stability is therefore a main topic of research. In [4], the geometric stability condition was formulated, and found to be a necessary condition for stability of the split field PML. In [5], it was proved to be necessary also for stability of a modal PML, even though the complex frequency shift had a stabilizing effect.

The aim of this paper is to construct efficient layers based on the second order equations, for all materials, and also those violating the geometric stability condition. The PML equations are derived using a complex coordinate stretching technique, [6, 17]. We include a grid stretching parameter and a complex frequency shift. One advantage of this approach is that we can choose auxiliary variables so that the resulting system is strongly hyperbolic.

In computations using standard second order central finite differences, our PML behaves dramatically better than the corresponding first order PMLs, for materials where the geometric stability condition is violated. In many cases no growth is seen in the computation even at very late times. A large part of the paper is dedicated to understanding why our PML behaves in this stable way, and how the stable behavior can be enhanced.

We start by applying a standard perturbation analysis to our PML at constant coefficients. The result is that our PML suffers from the same high frequency instability as the above mentioned first order PMLs for the geometric stability violating materials. From the analysis we know that the instability appears only at sufficiently high spatial frequencies. If these frequencies are not well resolved, the discrete behavior may be completely different. A straight forward computation of the temporal eigenvalues corresponding to the discrete spatial operator in a constant coefficient setting shows that if unstable modes are not well resolved, they are in fact stable in the discrete setting. We have investigated how the grid stretching parameter can be used to enhance this effect.

A second reason is the stabilizing effect of corner regions. When the layers are used as boundary closures completely surrounding a domain there are usually corner regions. We use the same perturbation technique as above applied to a constant coefficient corner problem, and find that our PML is significantly more stable in the corner region. In computations we have observed that the bulk of an unstable mode typically is localized to part of the layer and propagates tangentially while the amplitude grows. Eventually the bulk of the unstable mode moves into a corner region and is damped.

The paper is organized as follows. In Section 2 we introduce the elastic wave equa-