

The Ultra Weak Variational Formulation Using Bessel Basis Functions

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Abstract. We investigate the ultra weak variational formulation (UWVF) of the 2-D Helmholtz equation using a new choice of basis functions. Traditionally the UWVF basis functions are chosen to be plane waves. Here, we instead use first kind Bessel functions. We compare the performance of the two bases. Moreover, we show that it is possible to use coupled plane wave and Bessel bases in the same mesh. As test cases we shall consider propagating plane and evanescent waves in a rectangular domain and a singular 2-D Helmholtz problem in an L -shaped domain.

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1 Introduction

Interest in methods which use non-polynomial basis functions has increased recently because of their potential computational efficiency, especially, for higher frequency problems. Nevertheless, the numerical approximation of wave propagation problems still remains challenging and time consuming. One of these non-polynomial basis functions methods, herein studied, is called the ultra weak variational formulation (UWVF). The UWVF was first introduced and analyzed by Cessenat and Després [6–8] for the Helmholtz equation and Maxwell equations. To date the UWVF has been applied in many physical problems, for example, audio acoustics [21], ultrasound acoustics [17], electromagnetics [18], optoelectronics [22] and elasticity [19].

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The UWVF is a volume based method and uses triangular or tetrahedral meshes similar to those used in the finite element method (FEM). However, in the UWVF, solutions are computed on element edges in the 2-D case (and on element faces in the 3-D case). When simulating at high frequencies, the UWVF is more efficient than traditional FEM, see, for example, [8].

The original UWVF uses plane waves as a basis functions in part because integrals encountered in the method can be computed efficiently in closed form. Other non-polynomial basis methods include the discontinuous enrichment method (DEM) [9], partition of unity finite element method (PUFEM) [23], least squares method (LSM) [24] and discontinuous Galerkin method (DGM) [10]. In fact, it has been recently shown (see [11, 18]) that the UWVF is a special form of an upwind DGM. Some of these methods (PUFEM, LSM, UWVF and DGM) have also been compared to each other for 2-D Helmholtz problems.

Namely, the UWVF and PUFEM were compared to each other in [16] where the authors showed that the UWVF worked better at high frequencies and PUFEM at low frequencies. It has also been shown in [13] that at high frequencies the UWVF and DGM provide better accuracy than LSM while at low frequencies all three methods have similar errors. The use of Bessel basis functions has been studied in the DGM applying them to ultrasound and electromagnetic problems [2], and in the LSM [24].

In the UWVF, accuracy can be improved by refining the grid and/or using more basis functions on an element. However, if the elements are small compared to the wavelength, or too many basis functions are used, or when simulating at low frequencies, ill-conditioning may occur. One possible technique for improving the UWVF at low frequencies is to use a hybridized mixed FEM introduced in reference [25]. However, motivated by the work of Gittelsohn, Hiptmair and Perugia [14] and Barnett and Betcke [3], Bessel basis functions are considered in this paper. The goal is to improve the UWVF for problems with small elements or singularities. Unfortunately, for the Bessel basis, the UWVF-integrals must be computed using quadratures. On the other hand, Barnett and Betcke [3] reported interesting results for problems with singularities using only Bessel basis functions. Our intention is to use plane wave basis and Bessel basis functions in the UWVF.

We shall study 2-D Helmholtz problems. One of our model problems will be a singular 2-D Helmholtz problem on an L -shaped domain that was also a model problem in [12, 16]. In addition, we shall study propagating plane and evanescent waves in a rectangular domain.

This paper is organized as follows. In Section 2 the Helmholtz problem and UWVF are introduced. In Section 3 the different choices of basis functions are given. Section 4 is devoted to numerical simulations and is divided in two parts: the first problem studies a propagating plane wave and evanescent wave in a rectangular domain and second studies a singular 2-D Helmholtz problem. Finally, we draw conclusions in Section 5.