Commun. Comput. Phys.
doi: 10.4208/cicp.081209.070710s

Vol. **11**, No. 2, pp. 335-350
February 2012

A Modified Discontinuous Galerkin Method for Solving Efficiently Helmholtz Problems

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Received 8 December 2009; Accepted (in revised version) 7 July 2010

Available online 24 October 2011

Abstract. A new solution methodology is proposed for solving efficiently Helmholtz problems. The proposed method falls in the category of the discontinuous Galerkin methods. However, unlike the existing solution methodologies, this method requires solving (a) *well-posed* local problems to determine the primal variable, and (b) a global positive *semi-definite Hermitian* system to evaluate the Lagrange multiplier needed to restore the continuity across the element edges. Illustrative numerical results obtained for two-dimensional interior Helmholtz problems are presented to assess the accuracy and the stability of the proposed solution methodology.

AMS subject classifications: 35J05, 65N30, 49M27, 35L05, 93E24

Key words: Helmholtz equation, discontinuous Galerkin, plane waves, Lagrange multipliers, inf-sup condition, waveguide problems.

1 Introduction

The Helmholtz equation belongs to the classical equations of mathematical physics that are well understood from a mathematical view point. However, the numerical approximation of the solution is still a challenging problem in spite the tremendous progress made during the past fifty years (see, for example, the recent monograph [18] and the

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references therein). Indeed, the standard finite element method (FEM) is not well suited for solving Helmholtz problems in the mid- and high-frequency regime because of the quasi-optimality constant which grows with the wavenumber k, as explained in details in [5]. In order to maintain a certain level of accuracy while increasing the frequency, a mesh refinement is required and/or higher order FEM are used, leading to a prohibitive computational cost for high wavenumbers.

In response to this challenge, alternative techniques were proposed. Numerous of these approaches use the plane waves, since they are expected to better approximate highly oscillating waves [4,6–12,19,20,23]. In the discontinuous Galerkin method (DGM) designed by Farhat *et al.* and presented in a series of papers [8–10], the solution is approximated at the element mesh level using a superposition of plane waves which results in a discontinuous solution along interior boundaries of the mesh. The continuity is then restored weakly with Lagrange multipliers. The rectangular and quadrilateral elements constructed in [8–10] clearly outperform the standard Galerkin FEM. For example, for $ka \geq 10$ and for a fixed level of accuracy, the so-called R-4-1 element reduces the total number of degrees of freedom (dofs) required by the $Q1$ finite element by a factor greater or equal to five. Similar results are obtained for the R-8-2a and R-8-2b elements when compared to the $Q2$ element, and for Q-16-4 and Q-32-8 when compared to the $Q4$ element. In spite of this impressive performance, the DGM has three important drawbacks. First, the method has to satisfy an *inf-sup* condition which is translated, in practice, as a compatibility requirement: the number of dofs of the Lagrange multiplier (corresponding to the dual variable) and of the field (the primal variable) cannot be chosen arbitrarily. The problem here is that there is no theoretical result on how to satisfy this compatibility requirement, except for the simple case of R-4-1 element (see [2]). Hence, for other elements, the existing choices are based on numerical experiments only. The second major issue with the DGM is that it becomes unstable as we refine the mesh [1]. Such instabilities occur because of the singularity of the local problems and, to some extent, to the loss of the linear independence of the plane waves as the step size mesh discretization tends to zero. The latter affects dramatically the stability of the global system due to its ill-conditioning nature. Finally, the DGM exhibits a loss of accuracy for unstructured mesh [9].

We propose a new solution methodology for Helmholtz problems, that falls in the category of discontinuous Galerkin methods. The proposed formulation distinguishes itself from existing procedures by the *well-posed* character of the local problems and by the resulting global system which is associated with a positive *semi-definite Hermitian* matrix. More specifically, the computation domain is subdivided in quadrilateral- or triangular-shaped elements. The solution is approximated, at the element level, by a superposition of plane waves that are solution of the Helmholtz equation. The continuity of the solution at the interior interfaces of the elements is then enforced by Lagrange multipliers. Unlike the DGM, the proposed method does not require the continuity of the normal derivative. Consequently, Lagrange multipliers are introduced to restore, in a weak sense, the continuity of both the field and its normal derivative across interior boundaries of the mesh.