Commun. Comput. Phys.
doi: 10.4208/cicp.231109.090710s

Vol. **11**, No. 2, pp. 303-318
February 2012

Operator Factorization for Multiple-Scattering Problems and an Application to Periodic Media

J. Coatléven* and P. Joly

POems Project Team, UMR ENSTA/CNRS/INRIA, Inria Rocquencourt, 78153 Le Chesnay Cedex, France.

Received 23 November 2009; Accepted (in revised version) 9 July 2010

Available online 24 October 2011

Abstract. This work concerns multiple-scattering problems for time-harmonic equations in a reference generic media. We consider scatterers that can be sources, obstacles or compact perturbations of the reference media. Our aim is to restrict the computational domain to small compact domains containing the scatterers. We use Robin-to-Robin (RtR) operators (in the most general case) to express boundary conditions for the interior problem. We show that one can always factorize the RtR map using only operators defined using single-scatterer problems. This factorization is based on a decomposition of the diffracted field, on the whole domain where it is defined. Assuming that there exists a good method for solving single-scatterer problems, it then gives a convenient way to compute RtR maps for a random number of scatterers.

AMS subject classifications: 35C15, 35J05, 35Q60, 74J20

Key words: Multiple-scattering, harmonic wave equation, exact boundary conditions, periodic media.

1 Introduction

The present study has been motivated by the computation of wave propagation in locally perturbed periodic media. The typical application is numerical modeling of photonic crystals (see e.g. [7, 10]). The starting point is the method developed in [5] and [4] for the treatment of one small local defect (typically localized in one or a few periodicity cells). Our objective in this paper is to treat the case of several well separated defects of this nature by exploiting the existing method for one single defect. This problem enters the more general framework of multiple-scattering (see for instance [8]). The outline of the article is the following: in Section 2 we present our model problem in the more general case of a

*Corresponding author. *Email addresses:* `julien.coatleven@inria.fr` (J. Coatléven), `patrick.joly@inria.fr` (P. Joly)

propagation medium which is a perturbation of a given reference medium (which will be the periodic medium in the application) and we present our main objective: determine a transparent "Robin-to-Robin boundary condition" to reduce the effective computation to a small neighbourhood of the local defect. In Section 3 we present a method of decomposition of the solution of the multiple scattering problem into a sum of solution of single-scattering problems. This is the basis of the factorization of the transparent operator as a product of two operators that can be determined by solving only single-scattering problems (Section 4). Finally, in Section 5 we present numerical results obtained by applying this method to the case of a periodic reference medium.

2 Model problem and objectives

2.1 Setting of the problem

Let (Ω_j), $1\leq j\leq N$, be a family of bounded, connected, open sets of $\Omega=\mathbb{R}^n$, with at least a Lipschitz boundary. The domain $\Omega_{int}:=\bigcup_j\Omega_j$ will play the role of the desired computational domain (with the subscript "int" standing for interior). They are supposed to contain the support of the sources and the regions where the true propagation domain differs from a reference media which is supposed to have a simpler structure. When $N=1$ or equivalently when only one of the Ω_j's exists, one can speak of a single-scattering problem. To be more precise, we wish to solve the following Helmholtz equation in Ω:

Find u in $H^1(\Omega)$ such that:

$$-\Delta u-n^2(\omega^2+i\varepsilon\omega)u=f,\quad \text{in } \Omega, \tag{2.1}$$

where $\varepsilon>0$ represents the absorption of the medium (possibly arbitrary small).

We suppose that the functions $f\in L^2(\Omega)$ $n\in L^\infty(\Omega)$ are such that:

- $supp\ f\subset\Omega_{int}$, so we will write $f_j\in L^2(\Omega)=\chi_{\Omega_j}f$ such that $supp\ f_j\subset\Omega_j$.
- There exists a reference function $n_{ref}\in L^\infty(\Omega)$, such that $supp\ (n^2-n_{ref}^2)\subset\Omega_{int}$.
- For almost every $x\in\Omega$, $0<n_-\leq n(x)\leq n_+$, $0<n_-\leq n_{ref}(x)\leq n_+$.

With these technical hypothesis, the problem is well posed by Lax-Milgram's theorem.

Remark 2.1. In order to avoid lengthy notations, we have omitted on purpose the case where the domain Ω contains obstacles where the solution is not defined. This case can of course be treated using the method we will present here, provided that the Ω_j's are such that they contain all these obstacles.

It is interesting to remark that the model problem (2.1) takes into account "real" scattering problems, i.e. problems of the form: Find the total field u_{tot} such that $u_{tot}=u_{inc}+u_{diff}$, where u_{inc} is the "incident" field, u_{diff} the "diffracted" field and

$$-\Delta u_{tot}-n^2(\omega^2+i\varepsilon\omega)u_{tot}=0,\quad \text{in } \Omega,$$