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Multiscale Nanorod Metamaterials and Realizable Permittivity Tensors

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Abstract. Our aim is to evidence new 3D composite diffractive structures whose effective permittivity tensor can exhibit very large positive or negative real eigenvalues. We use a reiterated homogenization procedure in which the first step consists in considering a bounded obstacle made of periodically disposed parallel high conducting metallic fibers of finite length and very thin cross section. As shown in [2], the resulting constitutive law is non-local. Then by reproducing periodically the same kind of obstacle at small scale, we obtain a local effective law described by a permittivity tensor that we make explicit as a function of the frequency. Due to internal resonances, the eigenvalues of this tensor have real part that change of sign and are possibly very large within some range of frequencies. Numerical simulations are shown.

AMS subject classifications: 35B27, 35Q60, 35Q61, 78M35, 78M40

Key words: Homogenization, photonic crystals, metamaterials, micro-resonators, effective tensors, non local effects.

1 Introduction

In recent years, the advent of negative index metamaterials and composites has led to increased interest in effective medium theories. In particular there is presently intense activity in constructing artificial photonic crystals made of metallo-dielectric inclusions with the goal of reaching negative bulk electric or magnetic response. An excellent example is the wire medium studied by Pendry in 1996 [7] where it is suggested that high conductivity fibers occupying a very small volume fraction could produce negative permittivity. A rigorous proof of this based on homogenization techniques appeared in [8] where, instead of letting the wavelength λ tend to infinity as customary in effective medium theories, we keep it constant (assuming a time dependence of the electromagnetic field $\exp(-i\omega t)$) and let other geometrical parameters (as the period) tend to zero.

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The advantage of this approach is that it leaves us the possibility of keeping fixed some of the geometrical parameters, in particular the subset occupied by the composite obstacle illuminated by an incident wave of a given frequency ω.

It is important to notice that in [7, 8, 12], the analysis of the wire medium problem is considerably simplified by assuming infinitely long fibers: the incident wave can be then decomposed so that reduction to polarized fields is possible and mathematically the problem is reduced to solving Helmholtz equations in the plane with suitable transmission conditions on the boundary of the wire cross sections.

In case of an $\mathbf{e_3}$-polarized electric field and suitably scaled period, filling ratio of fibers and conductivity, it is shown in [8] that the medium is characterized in the quasi static regime (i.e., for $d \ll \lambda$) by an effective relative permittivity (in the $\mathbf{e_3}$-direction) of the form

$$\varepsilon_{33}^{\mathrm{eff}}(\omega) = 1 - \frac{\omega_c^2}{\omega^2}, \tag{1.1}$$

where ω_c represents the so called *cut-off* frequency. Furthermore the asymptotic formula (1.1) turns out to be very accurate even for small ratio $\lambda/d < 10$ as shown in Fig. 3.

Although this result pushes towards a mathematical foundation for the realizability of effective media with a negative bulk permittivity, we have however to be very careful. As will be seen later the model of a *finite* metallic wire medium ("bed-of-nails" structure) is more sophisticated and, unlike common practice in much of the metamaterials literature, it is not correct to assume that the finite structure behaves like an equivalent homogeneous medium of the same size characterized by the same constitutive relation as for the *infinite* medium, even if additional boundary conditions are imposed in order to account for the finiteness of the structure. It turns out that the extreme nonlocality of the initial structure implies automatically that the effective limit law is *non local* as well. In [2] the scatter consists of a domain of $\mathbb{R}^3$ filled by a periodic array of $\mathbf{e_3}$-parallel metallic fibers of finite length L as depicted in Fig. 1. By using a two-scale renormalization approach, we have shown that the effective constitutive equation between the displacement vector D and the electric field E involves a long range interaction kernel:

$$D(x) := \varepsilon_0 \Big(E(x) - 2\pi\gamma \mathbf{e_3} \int_{-\frac{L}{2}}^{\frac{L}{2}} g(\omega, s, x_3) E_3(x_1, x_2, s)\, ds \Big). \tag{1.2}$$

The same kind of non local constitutive law arises if we alternatively consider metallic arrays of parallel fibers disposed simultaneously in three orthogonal directions. It becomes therefore clear that the validity of an effective relation such as in (1.1) cannot be extended to a full 3D-setting, as far as composite made of metallic inclusions are concerned. Hence arises naturally the question of realizability of *a full 3D local negative effective permittivity tensor*.

In this paper we present a rigorous proof for the realizability at a given frequency of any real symmetric permittivity tensor (this includes the negative scalar ones). This result already announced in [3] is to our knowledge the first one obtained in the context of the diffraction of an electromagnetic wave by a finite obstacle in $\mathbb{R}^3$.