Commun. Comput. Phys.
doi: 10.4208/cicp.111209.160610s

Vol. **11**, No. 2, pp. 367-382
February 2012

Antiplane Wave Scattering from a Cylindrical Void in a Pre-Stressed Incompressible Neo-Hookean Material

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Received 11 December 2009; Accepted (in revised version) 16 June 2010

Available online 24 October 2011

Abstract. An isolated cylindrical void is located inside an incompressible nonlinear-elastic medium whose constitutive behaviour is governed by a neo-Hookean strain energy function. In-plane hydrostatic pressure is applied in the far-field so that the void changes its radius and an inhomogeneous region of deformation arises in the vicinity of the void. We consider scattering from the void in the deformed configuration due to an incident field (of small amplitude) generated by a horizontally polarized shear (SH) line source, a distance r_0 (R_0) away from the centre of the void in the deformed (undeformed) configuration. We show that the scattering coefficients of this scattered field are unaffected by the pre-stress (initial deformation). In particular, they depend not on the deformed void radius a or distance r_0, but instead on the *original* void size A and *original* distance R_0.

AMS subject classifications: 74B15, 74B20, 74J20, 74Q15

Key words: Pre-stress, incompressible, neo-Hookean, rubber, incremental SH waves, scattering, line source.

1 Introduction

The influence of nonlinear pre-stress on subsequent incremental linear wave propagation in elastic media has been studied in detail over the past few decades using the so-called theory of *small-on-large* [9, 15] where a linearization is performed about the nonlinear equilibrium state in order to determine the wave propagation characteristics of the pre-stressed medium. To the authors' knowledge, in the literature interest has centered exclusively on the influence of *homogeneous* stretch distributions (and hence induced anisotropy) on subsequent wave propagation, see e.g., [5, 10]. When the medium

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in question is *inhomogeneous* (for example a fibre-reinforced, or particulate composite material, where the host phase is nonlinear-elastic) then pre-stress will almost always lead to non-homogeneous stretch distributions, except in very special cases (see e.g., [17]). Degtyar et al. [6] analysed the case of stressed composites where residual stresses occur in the vicinity of inhomogeneities, in the context of linear elasticity. The interest was to determine how the presence of linear elastic residual stress affects the subsequent wave propagation through the inhomogeneous material and hence derive the associated effective wavenumber in the pre-stressed state. This work utilized the self consistent scattering formulation of the effective wavenumber derived for a single inclusion by Yang and Mal [19] which itself is a modification of the Waterman and Truell theory [18]. This modified formulation of multiple scattering theory was used since the stress distribution in the neighbourhood of an inhomogeneity was approximated as piecewise constant (i.e., a multi-layered cylindrical inclusion). We emphasize that this analysis took place in the linear elastic (small displacement gradient) regime. Of great interest however is how an initial *nonlinear* pre-stress affects subsequent multiple scattered wave fields which propagate through inhomogeneous media. In the low frequency regime this information would allow us to determine the incremental behaviour of the material and hence the effective homogenized properties. Recently, the so-called second-order homogenization method was applied to determine such homogenized behaviour in the static regime for nonlinear fibre reinforced composites [3]. The composite cylinder assemblage model for nonlinear composites has also been employed in this context [4]. Nonlinear pre-stress can be extremely useful in practice, allowing us to *tune* materials in order to permit or restrict the propagation of waves in specific frequency ranges. This property was described by Parnell [17] and discussed further in subsequent articles in different contexts [2,7].

Multiple scattering in the (unstressed) linear elastic quasi-static regime is relatively well understood [12] and much work has centered on the derivation of the effective wave number (and resulting effective elastic properties) given a random distribution of inhomogeneities. However, if the host medium is nonlinear-elastic (e.g., rubber) and we are interested in the multiple scattering problem in the pre-stressed configuration, a canonical single scattering problem of central importance, which must be studied before we can solve the multiple scattering problem, is the following: *How does an isolated inclusion embedded in a nonlinear-elastic pre-stressed host medium (where inhomogeneous deformation or stress is present), scatter incoming elastic waves?* To the authors' knowledge, no such problems of this type (i.e., incorporating inhomogeneous fields) have been solved before in the literature. In this article we shall consider the problem of antiplane elastic or horizontally-polarized shear (SH) wave scattering from a cylindrical void embedded in an incompressible host medium which is capable of finite deformation and is neo-Hookean in its constitutive behaviour. This problem in the context of no pre-stress is discussed on pp. 123 of [12] and pp. 208 of [16]. A related pre-stress problem is studied in [11]. However, in [11] the pre-stress was assumed to be uniform, i.e., *all* stretch distributions in the host domain are homogeneous. This is a simpler problem than that to be discussed in the present article since it changes only the induced anisotropy of the host