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# Elements of Mathematical Foundations for Numerical Approaches for Weakly Random Homogenization Problems

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**Abstract.** This work is a follow-up to our previous work [2]. It extends and complements, both theoretically and experimentally, the results presented there. Under consideration is the homogenization of a model of a weakly random heterogeneous material. The material consists of a reference periodic material randomly perturbed by another periodic material, so that its homogenized behavior is close to that of the reference material. We consider laws for the random perturbations more general than in [2]. We prove the validity of an asymptotic expansion in a certain class of settings. We also extend the formal approach introduced in [2]. Our perturbative approach shares common features with a defect-type theory of solid state physics. The computational efficiency of the approach is demonstrated.

**AMS subject classifications**: 35B27, 35J15, 35R60, 82D30

**Key words**: Homogenization, random media, defects.

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## 1 Introduction

Our purpose is to follow up on our previous study [2]. Let us recall, for consistency, that we consider homogenization for the following elliptic problem

$$\begin{cases} -\operatorname{div}\left(\left(A_{per}\left(\dfrac{x}{\epsilon}\right)+b_\eta\left(\dfrac{x}{\epsilon},\omega\right)C_{per}\left(\dfrac{x}{\epsilon}\right)\right)\nabla u_\epsilon\right)=f(x), & \text{in } \mathcal{D}\subset\mathbb{R}^d, \\ u_\epsilon=0, & \text{on } \partial\mathcal{D}, \end{cases} \tag{1.1}$$

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where the tensor $A_{per}$ models a reference $\mathbb{Z}^d$-periodic material which is randomly perturbed by the $\mathbb{Z}^d$-periodic tensor $C_{per}$, the stochastic nature of the problem being encoded in the stationary ergodic scalar field $b_\eta$ (the latter getting small when $\eta$ vanishes). We have studied in [2] the case of a perturbation that has a Bernoulli law with parameter $\eta$, meaning that $b_\eta$ is equal to 1 with probability $\eta$ and 0 with probability $1-\eta$. In the present work, we address more general laws. The common setting is that all the perturbations we consider are, to some extent, rare events which, although rare, modify the homogenized properties of the material. Our approach is a perturbative approach, and consists in approximating the stochastic homogenization problem for

$$A_\eta(x,\omega) = A_{per}(x) + b_\eta(x,\omega) C_{per}$$

using the periodic homogenization problem for $A_{per}$. In short, let us say that our main contribution is to derive an expansion

$$A_\eta^* = A_{per}^* + \eta \bar{A}_1^* + \eta^2 \bar{A}_2^* + o(\eta^2), \tag{1.2}$$

where $A_\eta^*$ and $A_{per}^*$ are the homogenized tensors associated with $A_\eta$ and $A_{per}$ respectively, and the first and second-order corrections $\bar{A}_1^*$ and $\bar{A}_2^*$ can be, loosely speaking, computed in terms of the microscopic properties of $A_{per}$ and $C_{per}$ and the statistics of second order of the random field $b_\eta$. The formulation is made precise in [2] and in Sections 2 and 3 below.

Motivations behind setting (1.1), as well as a review of the mathematical literature on similar issues and a comprehensive bibliography, can be found in [2]. We complement our study of the perturbative approach introduced with [2] in two different directions.

In Section 2, we rigorously establish an asymptotic expansion of the homogenized tensor in a mathematical setting where our input parameter (the field $b_\eta$ in (1.1)) enjoys appropriate weak convergence properties, as $\eta$ vanishes, in a reflexive Banach space, namely a Lebesgue space $L^\infty(\mathcal{D}, L^p(\Omega))$ (with $p>1$). In such a setting, we are in position to rigorously prove a first order asymptotic expansion (announced in [3] and precisely stated in [3, Theorem 2.1] and Theorem 2.1 below) for the homogenization of $A_\eta$, using simple functional analysis techniques very similar to those exposed in [4]. In our Corollaries 2.1 and 2.2, the expansion is pushed to second order under additional assumptions.

Our aim in Section 3 is to further extend our formal theory of [2]. Recall that this formal theory, rather than manipulating the random field $b_\eta$ itself, consists in focusing on its law. We indeed assume that the image measure (the law) corresponding to the perturbation admits an expansion (see (3.3) below) with respect to $\eta$ in the sense of distributions. While [2] has only addressed the specific case of a Bernoulli law, we consider here more general laws and proceed with the same formal derivations. These derivations lead to a first-order correction $\bar{A}_1^*$ in (1.2) obtained as the limit when $N\to\infty$ of a sequence of tensors $A_1^{*,N}$ computed on the supercell $[-N/2,N/2]^d$. It is the purpose of Proposition 3.1 to prove the convergence of $A_1^{*,N}$. The second-order term $\bar{A}_2^*$ is likewise defined, in Proposition 3.2, as the limit of a sequence of tensors $A_2^{*,N}$ when $N\to\infty$. The proof of