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Extension of the High-Order Space-Time Discontinuous Galerkin Cell Vertex Scheme to Solve Time Dependent Diffusion Equations

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Abstract. In this paper, the high-order space-time discontinuous Galerkin cell vertex scheme (DG-CVS) developed by the authors for hyperbolic conservation laws is extended for time dependent diffusion equations. In the extension, the treatment of the diffusive flux is exactly the same as that for the advective flux. Thanks to the Riemann-solver-free and reconstruction-free features of DG-CVS, both the advective flux and the diffusive flux are evaluated using continuous information across the cell interface. As a result, the resulting formulation with diffusive fluxes present is still consistent and does not need any extra ad hoc techniques to cure the common "variational crime" problem when traditional DG methods are applied to diffusion problems. For this reason, DG-CVS is conceptually simpler than other existing DG-typed methods. The numerical tests demonstrate that the convergence order based on the L_2-norm is optimal, i.e. $\mathcal{O}(h^{p+1})$ for the solution and $\mathcal{O}(h^p)$ for the solution gradients, when the basis polynomials are of odd degrees. For even-degree polynomials, the convergence order is sub-optimal for the solution and optimal for the solution gradients. The same odd-even behaviour can also be seen in some other DG-typed methods.

AMS subject classifications: 65M99, 76M25

Key words: High-order method, space-time method, discontinuous Galerkin (DG) method, cell-vertex scheme (CVS), diffusion equations.

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1 Introduction

Recently, the authors developed a compact high order space-time scheme for hyperbolic conservation laws [1–4]. The method integrates the best features of the space-time Conservation Element/Solution Element (CE/SE) [5] method and the discontinuous Galerkin

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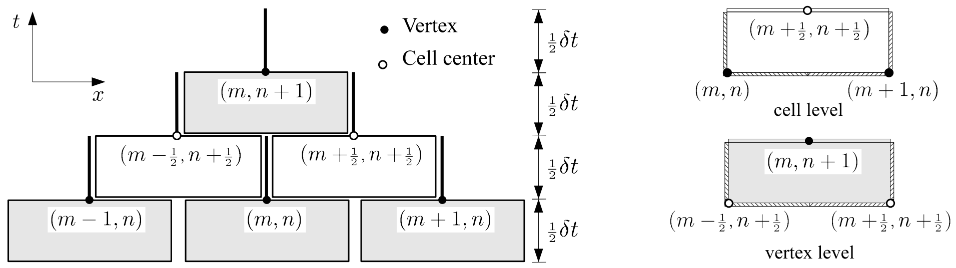

Figure 1: Solution elements (SEs) and conservation elements (CEs) in the x-t domain. Left: solution elements; right: conservation elements.

(DG) method. The core idea is to construct a staggered space-time mesh through alternate cell-centered CEs and vertex-centered CEs (cf. Fig. 1 (right)) within each time step. Inside each SE (cf. Fig. 1 (left)), the solution is approximated using high-order space-time DG basis polynomials. The space-time flux conservation is enforced inside each CE using the DG discretization. The solution is updated successively at the cell level and at the vertex level within each physical time step. For this reason and the method's DG ingredient, the method was named as the space-time discontinuous Galerkin cell-vertex scheme (DG-CVS) [3].

DG-CVS equally works on higher dimensions on arbitrary grids. Fig. 2 shows the conservation elements and solution elements on quadrilateral meshes and triangular meshes. Obviously, the definitions of CEs and SEs on higher dimensions are analogous to that for 1-D meshes (cf. Fig. 1). Fig. 3 demonstrates the resulting dual mesh at the cell level and the vertex level for both rectangular meshes and triangular meshes, respectively.

A summary of the main features of DG-CVS is given as follows:

- *Based on space-time formulation*. The space-time formulation is advantageous in handling moving boundary problems since it automatically satisfies the so-called Geometric Conservation Law.
- *High-order accuracy in both space and time*.
- *Riemann solver free*. In contrast to the traditional DG methods, DG-CVS does not need any numerical flux. The Riemann-solver-free feature offers two-fold advantages. First, this Riemann-solver-free approach eliminates some pathological behaviours associated with some Riemann solvers. Second, it is suitable for some hyperbolic PDE systems whose eigenstructures are not explicitly known.
- *Reconstruction free*. DG-CVS solves for the solution and its all spatial and temporal derivatives simultaneously at each space-time node, thus eliminating the need of reconstruction.
- *Suitable for arbitrary spatial meshes*.
- *Highly compact regardless of order of accuracy*.