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Runge-Kutta Discontinuous Galerkin Method Using WENO-Type Limiters: Three-Dimensional Unstructured Meshes

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Abstract. This paper further considers weighted essentially non-oscillatory (WENO) and Hermite weighted essentially non-oscillatory (HWENO) finite volume methods as limiters for Runge-Kutta discontinuous Galerkin (RKDG) methods to solve problems involving nonlinear hyperbolic conservation laws. The application discussed here is the solution of 3-D problems on unstructured meshes. Our numerical tests again demonstrate this is a robust and high order limiting procedure, which simultaneously achieves high order accuracy and sharp non-oscillatory shock transitions.

AMS subject classifications: 65M06, 65M99, 35L65

Key words: Runge-Kutta discontinuous Galerkin method, limiter, WENO, HWENO, high order limiting procedure.

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1 Introduction

Qiu *et al.* [16–18, 27, 28] have investigated weighted essentially non-oscillatory (WENO) and Hermite WENO (HWENO) finite volume methods as limiters for Runge-Kutta discontinuous Galerkin (RKDG) finite element methods [3–8], for the numerical solution of problems involving nonlinear hyperbolic conservation laws on structured and unstructured meshes. The goal is to construct a robust and high order limiting procedure that simultaneously achieves high order accuracy and sharp non-oscillatory shock transitions

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for the RKDG method, and in this paper we consider the solution of problems involving 3-D nonlinear hyperbolic conservation laws of form

$$\begin{cases} u_t+f(u)_x+g(u)_y+r(u)_z=0, \\ u(x,y,z,0)=u_0(x,y,z) \end{cases} \tag{1.1}$$

on 3-D unstructured meshes.

The WENO [9,11,12,14,25] and HWENO [16,18,26,27] schemes developed in recent years are a class of high order finite volume or finite difference schemes to numerically solve problems involving hyperbolic conservation laws, where both high order accuracy and essentially non-oscillatory shock transitions may be maintained. We have discussed third order finite volume WENO schemes in one space dimension [14], third and fifth order finite difference WENO schemes in various space dimensions with a general framework for the design of the smoothness indicators and nonlinear weights [12], and finite volume WENO schemes on structured and unstructured meshes [9,11,15,21,25]. The design of the WENO and also HWENO [16,18,26,27] schemes have been based on successful ENO schemes [10,23,24]. In both the ENO and WENO schemes, adaptive stencils were used in a reconstruction procedure based on local smoothness of the numerical solution, to automatically achieve high order accuracy and non-oscillatory behavior near discontinuities.

The first discontinuous Galerkin (DG) method was introduced in 1973 by Reed and Hill [19], for neutron transport described by steady state linear hyperbolic equations. A major development of the DG method was later carried out by Cockburn *et al.* in a series of papers [3–7]. They established a framework to readily solve problems involving *nonlinear* time-dependent hyperbolic conservation laws, via explicit nonlinearly stable high order Runge-Kutta time discretizations [23] and DG discretization in space, with exact or approximate Riemann solvers for interface fluxes and a total variation bounded (TVB) limiter [22] to achieve the non-oscillatory property for strong shocks. These schemes are now called RKDG methods.

To account for strong shocks in problems such as (1.1), an important component of a RKDG method is a nonlinear limiter to detect discontinuities and control any spurious oscillations that may arise nearby. Many such limiters have been used with RKDG methods. For example, the *minmod* TVB limiter [3–7] is a slope limiter using a technique borrowed from finite volume methodology, while a moment based limiter [1] and also an improved moment limiter [2] designed for discontinuous Galerkin methods use the moments of the numerical solution. However, these limiters tend to degrade accuracy when mistakenly used in smooth regions of the solution.

In [17], Qiu and Shu introduced the WENO methodology to provide limiters for the RKDG method on structured meshes, in the following way:

Step 1: First identify possible "troubled cells" – i.e. those cells that might need the limiting procedure.

Step 2: Replace the solution polynomials in these "troubled cells" by reconstructed