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A Comparison of Semi-Lagrangian and Lagrange-Galerkin hp-FEM Methods in Convection-Diffusion Problems

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Abstract. We perform a comparison in terms of accuracy and CPU time between second order BDF semi-Lagrangian and Lagrange-Galerkin schemes in combination with high order finite element method. The numerical results show that for polynomials of degree 2 semi-Lagrangian schemes are faster than Lagrange-Galerkin schemes for the same number of degrees of freedom, however, for the same level of accuracy both methods are about the same in terms of CPU time. For polynomials of degree larger than 2, Lagrange-Galerkin schemes behave better than semi-Lagrangian schemes in terms of both accuracy and CPU time; specially, for polynomials of degree 8 or larger. Also, we have performed tests on the parallelization of these schemes and the speed-up obtained is quasi-optimal even with more than 100 processors.

AMS subject classifications: 65M60, 65L60, 65N30, 65M25, 76M10, 76M25

Key words: Navier-Stokes equations, convection-diffusion equations, semi-Lagrangian, Lagrange-Galerkin, second order backward difference formula, hp-finite element method.

1 Introduction

This work is devoted to the study of convection dominated-diffusion problems and in particular the Navier-Stokes equations. Many efforts have been made by the numerical analysis community, and the Eulerian-Lagrangian approach has proven to produce very

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good results in many different situations. To fix ideas, we shall consider two models: (i) the prototype model of convection-diffusion equations

$$\begin{cases} \dfrac{\partial w}{\partial t}+\vec{u}\cdot\nabla w-\nu\Delta w=f, & \text{in } \Omega\times(0,T), \\ w|_{\partial\Omega}=g, \\ w(0)=w_0, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is an open bounded subset, $\vec{u}$ is the prescribed velocity vector, f denotes the forcing term, ν is the diffusion coefficient whereas w_0 and g denote the initial and Dirichlet boundary conditions respectively; (ii) the time-dependent incompressible Navier-Stokes equations

$$\begin{cases} \dfrac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}-\nu\Delta\vec{u}=-\nabla p+\vec{f}, & \text{in } \Omega\times(0,T), \\ \operatorname{div}\vec{u}=0, & \text{in } \Omega, \\ \vec{u}|_{\partial\Omega}=\vec{g}, \\ \vec{u}(0)=\vec{u}_0, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^2$ is an open bounded subset, $\vec{u}$ is the velocity of the fluid, p is the pressure, $\vec{f}$ is the forcing term, ν is the diffusion coefficient, $\vec{u}_0\in L^2(\Omega)$ is the prescribed initial condition and $\vec{g}$ is the boundary conditions. For the sake of simplicity, we shall consider homogeneous Dirichlet boundary conditions in the description of the numerical schemes.

In this article we propose a comparative study in terms of accuracy and CPU time of two of the most popular methods when using the Eulerian-Lagrangian approach, namely the Lagrange-Galerkin (or Characteristic Galerkin) and the semi-Lagrangian methods, because so far it is not clear which one of them should be used.

Both methods discretize the material derivative

$$\frac{Dw}{Dt}=\frac{\partial w}{\partial t}+\vec{u}\cdot\nabla w,$$

(in the Navier-Stokes equation, $w=\vec{u}$) along the characteristic curves $X(x,t_{n+1};t)$ defined as follows:

$$\begin{cases} X'(x,t_{n+1};t)=\vec{u}\big(X(x,t_{n+1};t),t\big), \\ X(x,t_{n+1};t_{n+1})=x, \end{cases} \tag{1.3}$$

where $t_{n-l}\le t\le t_{n+1}$, $l\in\mathbb{N}\cup\{0\}$, $x\in\Omega$, $t_j=j\Delta t$, and Δt the time step in the numerical scheme. Note that the characteristics curves are discretized backwards in time. This is one of the main advantages of the Eulerian-Lagrangian approach versus the Lagrangian one, which allows the mesh to follow the trajectories of the flow and so, the mesh is subject to large deformations. Another advantage of Eulerian-Lagrangian methods is that they have a large stability region, so that in the applications the time step can be chosen taking into account only accuracy considerations.