Commun. Comput. Phys.
doi: 10.4208/cicp.2009.09.067

Vol. 7, No. 4, pp. 738-758
April 2010

First-Order System Least-Squares Methods for a Flux Control Problem by the Stokes Flow

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Received 23 April 2009; Accepted (in revised version) 4 September 2009

Available online 28 October 2009

Abstract. This article deals with a first-order least-squares approach to the solution of an optimal control problem governed by Stokes equations. As with our earlier work on a velocity control by the Stokes flow in [S. Ryu, H.-C. Lee and S. D. Kim, SIAM J. Numer. Anal., 47 (2009), pp. 1524-1545], we recast the objective functional as a H^1 seminorm in the velocity control term. By introducing a *velocity-flux* variable and using the Lagrange multiplier rule, a first-order optimality system is obtained. We show that the least-squares principle based on L^2 norms applied to this system yields the optimal discretization error estimates for each variable in H^1 norm, including the velocity flux. For numerical tests, multigrid method is employed to the discrete algebraic system, so that the velocity and flux controls are obtained.

AMS subject classifications: 65M55, 65N30, 49J20, 49K20

Key words: Optimal control, optimization, least-squares finite element methods, Lagrange multiplier, Stokes equations.

1 Introduction

Optimal control problems governed by partial differential equations (PDEs) can be reduced to a system of coupled PDEs by the Lagrange multiplier method [13, 14, 18, 25]. Such system of coupled PDEs and optimal control problems have been interesting subjects because of not only their importance in the design process but also the needs of efficient and numerical methods for implementations.

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There have been considerable attentions in first-order system least-squares (FOSLS) approaches for Navier-Stokes or Stokes equations in many literatures [6,9,10,12,20–23]. These principles result in symmetric positive definite algebraic systems. Moreover, they enable us to avoid using the finite elements satisfying the inf-sup condition. Some applications of least-squares finite element methods to optimization problem have been previously discussed in [3–5,7,8,15,24,27]. In particular, [8] developed an abstract form of the FOSLS for optimal control problems governed by elliptic PDEs. Recently, [27] provides a nice synthesis of optimal control by Stokes flows and FOSLS principles by drawing on [8] and [12]. In [27], they only considered a velocity control using L^2 norms in the objective functional. In this work, as an improved version of [27], we consider a flux control using the cost functional which has the H^1 seminorm as a velocity control term.

From the Poincaré-Friedrichs inequality, the velocity control is also obtained by a flux control. In a different way with the one in [27], the Lagrange multiplier method is used after introducing a new variable $\mathbf{U}=\nabla\mathbf{u}^t$, so that the value $\mathbf{V}$ is a Lagrange multiplier and $\mathbf{V}\neq\nabla\mathbf{v}^t$.

The objective functional we consider is

$$\mathcal{J}(\mathbf{u},\mathbf{f})=\frac{1}{2}\int_\Omega|\mathbf{u}-\hat{\mathbf{u}}|^2+|\nabla\mathbf{u}^t-\nabla\hat{\mathbf{u}}^t|^2\,dx+\frac{\sigma}{2}\int_\Omega|\mathbf{f}|^2\,dx, \tag{1.1}$$

where $(\cdot)^t$ denotes the transpose, $\hat{\mathbf{u}}$ is the given target velocity, σ is a positive penalty parameter. The constraint is the Stokes equations such that

$$\begin{cases} -\nu\Delta\mathbf{u}+\nabla p=\mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u}=0 & \text{in } \Omega, \\ \mathbf{u}=\mathbf{0} & \text{on } \partial\Omega, \\ \int_\Omega p\,dx=0, \end{cases} \tag{1.2}$$

where $\mathbf{u}$ and p denote the velocity and pressure, respectively, ν the constant kinematic viscosity, and $\mathbf{f}$ the control function. Here $\Omega\subset\mathbb{R}^n(n=2 \text{ or } 3)$ is a bounded convex polyhedron or has $C^{1,1}$ boundary. The problem we study is to find an optimal state $(\mathbf{u},p)$ and an optimal control $\mathbf{f}$ which minimize the H^1-norm distance between $\mathbf{u}$ and $\hat{\mathbf{u}}$ satisfying the Stokes system (1.2). If Ω is connected and bounded at least in one direction and $\hat{\mathbf{u}}\in[H_0^1(\Omega)]^n$, then there exists a constant $C(\Omega)$ such that

$$\|\mathbf{u}-\hat{\mathbf{u}}\|<C(\Omega)\|\nabla\mathbf{u}^t-\nabla\hat{\mathbf{u}}^t\|, \tag{1.3}$$

by the Poincaré-Friedrichs inequality (see [17]). From (1.3), the objective functional (1.1) can be replaced by

$$\mathcal{J}(\mathbf{u},\mathbf{f})=\frac{1}{2}\int_\Omega|\nabla\mathbf{u}^t-\nabla\hat{\mathbf{u}}^t|^2\,dx+\frac{\sigma}{2}\int_\Omega|\mathbf{f}|^2\,dx. \tag{1.4}$$