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# High Order Compact Schemes in Projection Methods for Incompressible Viscous Flows

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**Abstract.** Within the projection schemes for the incompressible Navier-Stokes equations (namely "pressure-correction" method), we consider the simplest method (of order one in time) which takes into account the pressure in both steps of the splitting scheme. For this scheme, we construct, analyze and implement a new high order compact spatial approximation on nonstaggered grids. This approach yields a fourth order accuracy in space with an optimal treatment of the boundary conditions (without error on the velocity) which could be extended to more general splitting. We prove the unconditional stability of the associated Cauchy problem via von Neumann analysis. Then we carry out a normal mode analysis so as to obtain more precise results about the behavior of the numerical solutions. Finally we present detailed numerical tests for the Stokes and the Navier-Stokes equations (including the driven cavity benchmark) to illustrate the theoretical results.

**AMS subject classifications**: 65M06, 76D05

**Key words**: Incompressible Navier-Stokes, fractional step method, high order compact scheme, boundary conditions.

## 1 Introduction

For four decades, projection methods have been widely developed for the numerical simulation of unsteady viscous incompressible flows — Navier-Stokes equations with primitive variables: velocity, pressure. Within this class of methods detailed below, the main purpose of this paper is the construction, analysis and implementation of high order compact space approximations on nonstaggered grids.

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The survey paper of Guermond, Minev and Shen [32] gave recently a rather complete overview of projection methods for incompressible flows. Our approach belongs to the class of "pressure-correction" methods presented in [32] (Part 3). Projection or splitting methods for incompressible flows were independently introduced by Chorin [1] and Temam [2] forty years ago. They carried out the splitting of velocity and pressure for Navier-Stokes equations which yields independent systems of elliptic equations for the velocity and the pressure. These pioneering works [1, 2] have been published simultaneously with the fractional-step (or splitting) methods for multidimensional partial differential equations problems (see Yanenko [35]). However these splittings imply a decoupling of space variables for elliptic or parabolic problems (generalization of ADI methods [30, 31, 38]). For Navier-Stokes equations, due to the specificity of the pressure (non dynamic variable), the projection methods are not relevant of the fractional step methodology.

The Chorin-Temam projection method [1, 2] completely decouples the velocity and the pressure in two steps. The first step is a "dynamic" step, an intermediate velocity field is computed without taking the pressure into account. In the second step, the pressure is obtained from an elliptic problem including the intermediate velocity carry out in the first step. Through this choice of construction, this projection method is of order 1 independently of the approximation order of the time derivative [5].

The next stage in the development of projection methods begins with the paper of Goda [3] in 1979. The main objective, subsequently followed by Kim-Moin [4], Van Kan [25], was the construction of projection methods of order 2. Therefore the Chorin-Temam approach is modified by the introduction of the pressure gradient in the first step. In this paper we use the splitting of Goda [3]; however our construction and analyzes may be extended to other projection methods.

The major feature of this paper is the use of spatial compact approximations of order 4 in the framework of projection methods. The efficiency, the robustness, the easy treatment of boundary conditions prompt many authors [6, 9, 16, 17, 19, 33, 34], to develop compact difference schemes (nine point schemes for 2D problems) for Navier-Stokes equations in vorticity stream function formulation. We also refer to the papers of Ben-Artzi et al. [20, 21] devoted to a compact scheme for Navier-Stokes equations under this formulation using biharmonic operator. The development of approximation methods for the pressure velocity formulation is crucial because their 3D generalization is quite natural. Following previous works on projection methods [10–12, 15, 37], we use the normal mode analysis developed from the pioneering works of Godunov-Riabenkii, Kreiss et al. [7,29,36] devoted to hyperbolic problems. Karniadakis, Israeli and Orszag [5], Orszag, Israeli and Deville [23] are the first authors who have applied this analysis to projection methods. The principle of normal mode analysis is the comparison, using Fourier and Laplace transforms, of the solution modes of the differential problem with the solution modes of the semi-discretized problem and those of the numerical approximation. This approach, in simplified situations, allows some precise results about the behavior of the numerical solutions, the influence of the boundary conditions and the presence of numer-