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A Least-Squares/Fictitious Domain Method for Linear Elliptic Problems with Robin Boundary Conditions

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To the memory of David Gottlieb

Abstract. In this article, we discuss a least-squares/fictitious domain method for the solution of linear elliptic boundary value problems with Robin boundary conditions. Let Ω and ω be two bounded domains of $\mathbf{R}^d$ such that $\overline{\omega}\subset\Omega$. For a linear elliptic problem in $\Omega\backslash\overline{\omega}$ with Robin boundary condition on the boundary γ of ω, our goal here is to develop a fictitious domain method where one solves a variant of the original problem on the full Ω, followed by a well-chosen correction over ω. This method is of the virtual control type and relies on a least-squares formulation making the problem solvable by a conjugate gradient algorithm operating in a well chosen control space. Numerical results obtained when applying our method to the solution of two-dimensional elliptic and parabolic problems are given; they suggest optimal order of convergence.

AMS subject classifications: 65M85, 65N85, 76M10, 93E24

Key words: Least-Square methods, fictitious domain methods, finite element methods, Robin boundary conditions.

1 Introduction

Fictitious domain methods for the solution of partial differential equations are very useful methods for the solution of complicated problems. To the best of our knowledge, these methods have been introduced by Hyman [1] and further investigated by many authors;

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let us mention, among others, Saul'ev [2, 3] and Buzbee, Dorr, George and Golub [4]. In Glowinski, Pan and Periaux [5–7] and Glowinski, Pan, Kearsley and Periaux [8], fictitious domain methods were discussed for the solution of Dirichlet problems, the Dirichlet boundary condition being enforced as a side constraint, using a *boundary supported Lagrange multiplier*. A volume-supported Lagrange multiplier based fictitious domain method was introduced in Glowinski, Pan, Hesla, Joseph and Periaux in [9], the main motivation of these authors being the direct numerical simulation of particulate flow when the number of particles exceeds 10^3. Initially tested on particulate flow with spherical particles, the method discussed in [9] was generalized to situations involving particles with more complicated shapes, as shown for example in Pan, Glowinski and Galdi [10].

The main idea behind fictitious domain methods is to extend a problem initially posed on a geometrically complex shaped domain to a larger simpler domain; this provides two main advantages when constructing numerical schemes: (i) the extended domain is geometrically simpler and allows the use of fast solvers. (ii) The same fixed mesh can be used for the entire computation, eliminating thus the need for repeated re-meshing and projection. All the studies that we know of, concerning the application of fictitious domain methods to the simulation of particulate flow, consider no-slip boundary conditions at the interface between fluid and particles. There are situations however, in *micro-fluidics* for example, where a slip condition on the particle surface is more realistic than the no-slip one. If the no-slip boundary condition on the particle surface is replaced by the Navier slip boundary condition, the volume-supported Lagrange multiplier based fictitious domain methods discussed in [5–10], which rely on H^1-extensions, are not easy to generalize to the slip situation.

The main goal of the present article is to discuss the solution of linear elliptic boundary value problems with *Robin boundary conditions*; we see this as a first step to the construction of fictitious domain methods suited to slip boundary conditions. The method is of the *virtual control* type (in the sense of J. L. Lions; see [11]) and relies on a *least-squares* formulation making the problem solvable by a *conjugate gradient* algorithm operating in a well-chosen control space.

The formulation of the boundary value problems is given in Section 2. In Section 3, we describe a least-squares/fictitious domain method for the solution of linear elliptic problems with Robin boundary conditions. In Section 4, we discuss the conjugate gradient solution of the least-squares problems introduced in Section 3. The *finite element* implementation of the above methodology is discussed in Section 5. Finally, we present in Section 6 the results of numerical experiments; in particular, these results suggest optimal order of convergence for various norms of the approximation error.

A (brief) history of *fictitious domain methods* can be found in, e.g., [12, Chapter 8].

2 Formulation of the boundary value problems

Let Ω and ω be two bounded domains of $\mathbf{R}^d$, such that $d \geq 1$ and $\overline{\omega} \subset \Omega$ (see Fig. 1). We denote by Γ and γ the boundaries of Ω and ω, respectively.