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An FFT Based Fast Poisson Solver on Spherical Shells

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To the memory of David Gottlieb

Abstract. We present a fast Poisson solver on spherical shells. With a special change of variable, the radial part of the Laplacian transforms to a constant coefficient differential operator. As a result, the Fast Fourier Transform can be applied to solve the Poisson equation with $\mathcal{O}(N^3\log N)$ operations. Numerical examples have confirmed the accuracy and robustness of the new scheme.

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1 Introduction

The purpose of this paper is to propose a simple fast solver for the Poisson equation in a spherical shell

$$\begin{cases} \dfrac{\partial_\rho(\rho^2\partial_\rho u)}{\rho^2}+\dfrac{\partial_\theta(\sin\theta\partial_\theta u)}{\rho^2\sin\theta}+\dfrac{\partial_\phi^2 u}{\rho^2\sin^2\theta}=f, \quad \text{in } \Omega, \\ u|_{\rho=\rho_{\min}}=u^L(\theta,\phi), \\ u|_{\rho=\rho_{\max}}=u^R(\theta,\phi), \end{cases} \tag{1.1}$$

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where

$$\Omega=\Big\{\rho_{\min}<\rho<\rho_{\max},\ 0\le\theta\le\pi,\ 0\le\phi\le 2\pi\Big\}.$$

The Poisson equation in the spherical shell geometry is important in many geophysical and solar-physical applications [5,14,15].

Eq. (1.1) can be put in a more symmetric form

$$\begin{cases} \partial_\rho\left(\rho^2\partial_\rho\sin^2\theta\ u\right)+(\sin\theta\partial_\theta)^2u+\partial_\phi^2u=\rho^2(\sin^2\theta)f, & \text{in}\ \ \Omega,\\ u|_{\rho=\rho_{\min}}=u^L(\theta,\phi),\\ u|_{\rho=\rho_{\max}}=u^R(\theta,\phi). \end{cases} \tag{1.2}$$

In this symmetric form (1.2), one can apply Fast Fourier Transform to both the θ and ϕ derivatives (see Section 2 for details) to obtain optimal efficiency. The major obstacle for developing an overall fast solver is the radial derivatives which constitute a variable coefficient differential operator. The most popular approaches include Poisson solvers based on FFT in two directions or spherical harmonic functions which requires a Fast Legendre transform [1,4,6,7,9,12,13,16]. There are also other approaches using different sets of grids such as the Cubed Sphere grid [11] and the Yin-Yang grid [17].

In this paper, we propose a simple alternative, which provides a more accessible fast solver to (1.2) via FFT in all three variables. We propose the following simultaneous change of dependent and independent variables

$$s=\frac{\ln\rho-\ln\rho_{\min}}{\ln\rho_{\max}-\ln\rho_{\min}}, \tag{1.3a}$$

$$v=\sqrt{\rho}\,u. \tag{1.3b}$$

It is easy to see that, under the transformation (1.3), the Poisson equation (1.1) now takes the form

$$\sin^2\theta\Big(\alpha\partial_s^2-\frac{1}{4}\Big)v+(\sin\theta\partial_\theta)^2v+\partial_\phi^2v=g\equiv\rho^{\frac{5}{2}}\sin^2\theta f, \tag{1.4}$$

where

$$\alpha=(\ln\rho_{\max}-\ln\rho_{\min})^{-2}, \tag{1.5}$$

with boundary data

$$v|_{s=0}=v^L(\theta,\phi)\equiv\sqrt{\rho_{\min}}\ u^L(\theta,\phi), \tag{1.6a}$$

$$v|_{s=1}=v^R(\theta,\phi)\equiv\sqrt{\rho_{\max}}\ u^R(\theta,\phi). \tag{1.6b}$$

The significance of the transformation (1.3) is that the radial part now becomes a constant coefficient differential operator. As a consequence, the discretized operator for $(\alpha\partial_s^2-1/4)$ can be fast-diagonalized via FFT, resulting in an fast solver with total $\mathcal{O}(N^3\log N)$