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## Review Article

# A Review of David Gottlieb's Work on the Resolution of the Gibbs Phenomenon

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*To the memory of David Gottlieb*

**Abstract.** Given a piecewise smooth function, it is possible to construct a global expansion in some complete orthogonal basis, such as the Fourier basis. However, the local discontinuities of the function will destroy the convergence of global approximations, even in regions for which the underlying function is analytic. The global expansions are contaminated by the presence of a local discontinuity, and the result is that the partial sums are oscillatory and feature non-uniform convergence. This characteristic behavior is called the Gibbs phenomenon. However, David Gottlieb and Chi-Wang Shu showed that these slowly and non-uniformly convergent global approximations retain within them high order information which can be recovered with suitable post-processing. In this paper we review the history of the Gibbs phenomenon and the story of its resolution.

**AMS subject classifications**: 42A10, 41A10, 41A25

**Key words**: Gibbs phenomenon, post-processing, Galerkin approximation, collocation approximation, spectral methods, exponential accuracy.

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# 1 Introduction

The purpose of this paper is to review the history of the Gibbs phenomenon and the groundbreaking work of David Gottlieb, Chi-Wang Shu, and their co-workers, [16,25–30] on the resolution of the Gibbs phenomenon, for this special issue in memory of David Gottlieb.

To understand the Gibbs phenomenon, we begin with a classical problem: the square wave function,

$$f(x)=\begin{cases} -1, & -1\leq x<0,\\ 1, & 0\leq x\leq 1, \end{cases}$$

which can be written as a Fourier sine series

$$f(x)=\frac{4}{\pi}\sum_{j=odd}^{\infty}\frac{1}{j}\sin(j\pi x)=\frac{4}{\pi}\Big(\sin(\pi x)+\frac{1}{3}\sin(3\pi x)+\frac{1}{5}\sin(5\pi x)+\frac{1}{7}\sin(7\pi x)+\cdots\Big).$$

However, when we look at a Fourier partial sum

$$f_N(x)=\frac{4}{\pi}\sum_{j=odd}^{N}\frac{1}{j}\sin(j\pi x),$$

we observe that it is oscillatory, and that there is an overshoot and undershoot near the discontinuity and the boundaries (Fig. 1). As more terms are used, the overshoot and undershoot get closer to the discontinuity $x=0$ and boundaries $x=\pm 1$, but do not diminish. This indicates that the convergence of the series is not uniform, i.e., even though for each fixed $x$, the sequence of Fourier partial sums converges as $N\to\infty$ (pointwise convergence), the sequence does not converge as $x\to 0$ and $N\to\infty$ simultaneously. Furthermore, even the pointwise convergence is slow, due to the oscillations.

The convergence of a Fourier series to a discontinuous (or, equivalently, a non-periodic) function is non-uniform, the partial sums are oscillatory and the pointwise convergence is slow, even when we are looking at a point $x$ for which the function is continuous. The oscillatory behavior of the Fourier finite sums was first remarked upon by Wilbraham in 1848, and later, inspired by Josiah Willard Gibbs' spirited correspondence in NATURE in 1898 and 1899, called the Gibbs phenomenon. The Gibbs phenomenon is due to the fact that the local behavior of the function (i.e., a discontinuity