Commun. Comput. Phys.
doi: 10.4208/cicp.050309.110510a

Vol. **9**, No. 2, pp. 441-480
February 2011

A Compact High Order Space-Time Method for Conservation Laws

Shuangzhang Tu*, Gordon W. Skelton and Qing Pang

Department of Computer Engineering, Jackson State University, Jackson, MS 39217, USA.

Received 5 March 2009; Accepted (in revised version) 11 May 2010

Communicated by Chi-Wang Shu

Available online 27 August 2010

Abstract. This paper presents a novel high-order space-time method for hyperbolic conservation laws. Two important concepts, the staggered space-time mesh of the space-time conservation element/solution element (CE/SE) method and the local discontinuous basis functions of the space-time discontinuous Galerkin (DG) finite element method, are the two key ingredients of the new scheme. The staggered space-time mesh is constructed using the cell-vertex structure of the underlying spatial mesh. The universal definitions of CEs and SEs are independent of the underlying spatial mesh and thus suitable for arbitrarily unstructured meshes. The solution within each physical time step is updated alternately at the cell level and the vertex level. For this solution updating strategy and the DG ingredient, the new scheme here is termed as the discontinuous Galerkin cell-vertex scheme (DG-CVS). The high order of accuracy is achieved by employing high-order Taylor polynomials as the basis functions inside each SE. The present DG-CVS exhibits many advantageous features such as Riemann-solver-free, high-order accuracy, point-implicitness, compactness, and ease of handling boundary conditions. Several numerical tests including the scalar advection equations and compressible Euler equations will demonstrate the performance of the new method.

AMS subject classifications: 65M99, 76M25

Key words: High order method, space-time method, cell-vertex scheme (CVS), conservation laws.

1 Introduction

To numerically solve hyperbolic conservation laws, many methods including the classic finite difference and finite volume methods, discontinuous Galerkin (DG) method [1, 6,

*Corresponding author. *Email addresses:* `shuangzhang.tu@jsums.edu` (S. Tu), `gordon.skelton@jsums.edu` (G. W. Skelton), `qing.pang@jsums.edu` (Q. Pang)

9, 17] and the spectral volume method [31] use exact or approximate Riemann solvers to provide inter-cell fluxes. Since Riemann fluxes are based on the local characteristic structure of the governing equations, they are expected to provide the correct fluxes and stabilize the numerical methods. However, in the actual implementation, Riemann fluxes are dependent on trace values which are obtained numerically. Riemann fluxes computed this way may deviate from the actual physical values. This might explain why many (approximate) Riemann solvers capable of resolving contact discontinuities as well as shocks often exhibit some pathological phenomena among which the so-called carbuncle problem [10, 21, 24] is the most notorious one. Instead of finding the "cure" for such phenomena, some researchers resort to the so-called Riemann-solver-free approaches to avoid such phenomena. Examples include the space-time Conservation Element and Solution Element (CE/SE) method [3, 5], the Nessyahu-Tadmor (NT) scheme [19] and its improved variant [12, 15]. Indeed, these schemes are often referred to as central schemes in contrast to the Riemann-solver-based upwind ones. These central schemes are free of the carbuncle problem and produce entropy-satisfying solutions. In addition, since no Riemann solvers are involved in these central schemes, computation of numerical fluxes do not need the eigen-structure information of the system, which is attractive in governing equations where the eigen structure is not explicitly or easily known.

Numerical methods are required to be of high-resolution in both space and time such that the complex and possibly transient physical features in the simulated flow field are not overly smeared out during the long-time simulation. High-resolution means that the numerical dissipative error and dispersive error inherent to the scheme are small compared to the corresponding physical ones. Traditionally, the second-order accurate schemes were considered as high-resolution schemes. This was true when compared to the dissipative first order methods. Second order schemes have been extensively adopted in many commercial packages due to their simpleness and acceptable accuracy for many engineering applications. However, there are also many applications where second order methods are not adequate. For example, in the field of aeroacoustics, small disturbances (acoustics) often co-exist with strong discontinuities (e.g., shock waves), second order schemes tend to smear out the small disturbances while capturing strong shocks. Other examples include the wake of rotor blades and the flow field around flapping wings of Micro Aerial Vehicles (MAVs) where the flow is highly unsteady and vortex abundant. In this situation, numerical schemes of higher resolution are required to capture the transient vortices.

High resolution methods usually employ high-order (higher than second order) spatial and temporal discretization. Various high-order methods such as the WENO (weighted essentially nonoscillatory) [14] scheme, the discontinuous Galerkin (DG) finite element method [7], the spectral element method (SEM) [22] and the spectral volume method (SVM) [31] have been proposed in the literature and studied by many researchers. High resolution methods may not be of high order in terms of truncation error. An example is the aforementioned CE/SE method [3, 5] that yields high-resolution solutions though it is designed to be second order accurate in both space and time. Solvers