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The WASP Model: A Micro-Macro System of Wave-Schrödinger-Plasma Equations for Filamentation

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Abstract. In this paper, we model laser-gas interactions and propagation in some extreme regimes. After a mathematical study of a micro-macro Maxwell-Schrödinger model [1] for short, high-frequency and intense laser-gas interactions, we propose to improve this model by adding a plasma equation in order to precisely take into account free electron effects. We examine if such a model can predict and explain complex structures such as *filaments*, on a physical and numerical basis. In particular, we present in this paper a first numerical observation of nonlinear focusing effects using an *ab-initio* gas representation and linking our results with existing nonlinear models.

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1 Introduction

We have introduced previously a numerical micro-macro Maxwell-Schrödinger system for the modeling of intense, ultrashort and high frequency laser pulses propagating in dense gaseous media [1, 2]. Numerical simulations were presented in [3–5], where the coupling of the macroscopic Maxwell equations with many Time Dependent Schrödinger Equations (TDSEs), is introduced *via* the exact polarization, thus physically linking the microscopic and macroscopic scales. This numerical model was the first one to our

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knowledge, that takes into account ionization (short enough pulses) and high order harmonic generation [6,7] at the molecular scale *via* many TDSEs. Usual nonlinear macroscopic models such as Nonlinear Schrödinger Equations (NLS) determine the medium response using macroscopic perturbative expansions. Maxwell-Bloch's equations go beyond the perturbative approach but are restricted to the case of resonantly coupled radiation to a specific transition in the medium.

Consequently, Maxwell-Bloch's equations *a priori* cannot describe various effects we are interested in, such as multiphoton ionization, high order harmonic generation (up to ionization) and then filamentation (that necessitates the inclusion of plasma of free electron effect). The Maxwell-Bloch equations have been mathematically studied by Dumas [8] using some techniques initially introduced for ferromagnetic media [9]. We prove in this paper the existence of weak solutions for the Maxwell-Schrödinger system in a H_2^+-gas (extension to other gas is straightforward). In contrast to nonlinear Maxwell's equations (see Eqs. (27-29) in [10]) in our approach the nonlinearity appears via medium polarization, which couples the Maxwell and Schrödinger equations which themselves are linear.

The Maxwell equations we consider are linear with constant coefficient but with nonlinear source terms. TDSEs are studied using energy estimates, Grönwall's inequality, classical functional analysis inequalities (Cauchy-Schwarz, Hardy, etc.) and finally Leray-Schauder's fixed point theorem. Note that a fundamental lemma on TDSEs necessary for the proof can be derived from [11]. We also prove that the regularity of the initial data is conserved in time, which is also an important information from a numerical point of view (choice of the numerical method in particular).

After proving the existence of solution, we focus on particular dynamic solutions appearing in nonlinear media called *filaments*. These are defined in [10] as dynamic structures with an intense core, that is able to propagate over extended distances much larger than the typical diffraction length while keeping a narrow beam size without the help of any guiding mechanism. An exhaustive phenomenological and physical description of this phenomenon can be found in [10,12].

In this paper, we wish to establish whether the phenomenon of filamentation is properly predicted by our Maxwell-Schrödinger model. Our mode includes ionization properly for very short pulses. In order to describe filamentation for longer laser pulses, commonly used in the experiments, we modify the Maxwell-Schrödinger equations [1] into a so-called Wave-Schrödinger-Plasma (WASP) equations adding an evolution equation on free electrons in order to take precisely plasma effect into account (and the current density in Maxwell's equations). Some elements of proof that filament-like structures can exist for the WASP model are then given. However a scale transform will be ultimately necessary to link our model and results to experimental observations of filamentation over long distances.

We also formally derive from the WASP model some classical nonlinear Schrödinger (NLS) equations that have numerically generated filaments (see [12,13] for instance). Note that some theoretical arguments for proving the existence of filaments for NLS