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Numerical Resolution Near $t=0$ of Nonlinear Evolution Equations in the Presence of Corner Singularities in Space Dimension 1

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Dedicated to the memory of David Gottlieb

Abstract. The incompatibilities between the initial and boundary data will cause singularities at the time-space corners, which in turn adversely affect the accuracy of the numerical schemes used to compute the solutions. We study the corner singularity issue for nonlinear evolution equations in 1D, and propose two remedy procedures that effectively recover much of the accuracy of the numerical scheme in use. Applications of the remedy procedures to the 1D viscous Burgers equation, and to the 1D nonlinear reaction-diffusion equation are presented. The remedy procedures are applicable to other nonlinear diffusion equations as well.

AMS subject classifications: 35B65, 35K55, 35K57, 65M20, 65M60

Key words: Compatibility conditions, corner singularities, viscous Burgers equation, nonlinear convection diffusion equation, finite element methods.

1 Introduction

It is well known in the mathematics community that smooth boundary and initial data do not guarantee smooth solutions to the initial-boundary value problems of time-dependent PDEs. Even if the existence and uniqueness of the solution are proved and all the given data are as smooth as desired, then, in order for the solution to be smooth near $t=0$, it is necessary and sufficient that the boundary data and the initial data satisfy an infinite set of so-called compatibility conditions (see [9–15, 17] and below).

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When the boundary and initial data fail to satisfy some of the compatibility conditions, singularities may occur at the corner of the time and spatial axes. In some cases, for simple problems, this is not a critical issue, because the singularity is short-lived.

However in recent years the issue of the compatibility conditions for the initial boundary value problems of time-dependent PDEs started to receive attention in the numerical simulation community, because larger and more complex problems are handled thanks to the ever-growing computing power that is available. This produces a need to better understand what happens during the short initial period for certain physical processes. Boyd and Flyer [3] analyzed the connection between incompatibility and the rate of convergence of Chebyshev spectral series, and discussed the remedy procedure of smoothing the initial conditions. Flyer and Swarztrauber [8] studied the effect of the incompatibilities on the convergence rate of spectral and finite difference methods. Flyer and Fornberg [6] proposed a remedy procedure, based on the idea of singular corner functions, for the heat equation, and for variable coefficient convection-diffusion equations as well. Flyer and Fornberg [7] studied the corner basis functions for some dispersive equations. Bieniasz [2] modified the remedy procedure proposed by Flyer and Fornberg [6], and applied it to a diffusion-reaction system arising from electrochemistry.

In this article, we study from a numerical point of view, the compatibility issue for nonlinear diffusive PDEs. We shall first present our approach in details for the classical viscous Burgers equation in dimension one. However our approach does not depend on any particular property of the Burgers equation other than its diffusiveness. Hence we believe that it applies to other nonlinear diffusive equations as well. To demonstrate this, we shall also apply the approach to a 1D nonlinear reaction-diffusion equation.

For the Burgers equation the correction procedure proposed in [6] cannot be expected to work to its full strength, for two reasons. The first is that, for a nonlinear PDE, the singular corner functions, which satisfy the equation and also display corner singularities, are hard, if not impossible, to find. The second reason is that the superposition property does not hold for nonlinear equations, which means that even if in some rare cases we find the singular corner functions for the nonlinear equation, we cannot separate them from the solution without introducing singular terms into the equation. However, we speculate here that the singular corner functions, derived from the linearized Burgers equation, can be employed to remove the zero order incompatibility (see Section 2.2). What is less obvious is that the singular corner functions can also be employed to remove higher order singularities (see Section 2.3). However, as has been said, we cannot avoid introducing singular terms into the equation. To overcome the difficulty associated with singular terms in the equation we choose the Galerkin finite element method (FEM) as the means for constructing the numerical scheme for the equation, because the Galerkin FEM is based on the weak formulation of the PDE, and hence is potentially more tolerant of the singularities in the equation. The numerical results confirm the effectiveness of the correction procedures that we propose.

Flyer et al. [6], Bieniasz [2] and the current study all seek the solutions of the target