

A Well-Balanced and Non-Negative Numerical Scheme for Solving the Integrated Shallow Water and Solute Transport Equations

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Abstract. Based on the recent development in shallow flow modelling, this paper presents a finite volume Godunov-type model for solving a 4×4 hyperbolic matrix system of conservation laws that comprise the shallow water and depth-averaged solute transport equations. The adopted governing equations are derived to preserve exactly the solution of lake at rest so that no special numerical technique is necessary in order to construct a well-balanced scheme. The HLLC approximate Riemann solver is used to evaluate the interface fluxes. Second-order accuracy is achieved using the MUSCL slope limited linear reconstruction together with a Runge-Kutta time integration method. The model is validated against several benchmark tests and the results are in excellent agreement with analytical solutions or other published numerical predictions.

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1 Introduction

Solute transport is a common process that may take place in rivers, lakes and estuarine and coastal areas where the flows have horizontal dimensions much larger than their vertical extent (shallow flows). It may be closely related to the water quality in these shallow water bodies and have great impacts on the local environment and ecosystem. It may also cause potential risk on public health and local economy, e.g. when it is associated with an urban flood event. To understand the solute transport process in a shallow flow is thus of fundamental and practical importance to hydraulic and environmental

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engineering. In this work, we consider the passive solute transport driven by shallow flows, where 'passive' essentially means that the solute particles (or concentration) are drifted by the fluid velocity and their feedback to the flow is negligible. This assumption is realistic for most of the engineering problems with low solute concentration.

In practice, it is common to assume that shallow flows are predominantly horizontal with hydrostatic pressure distribution so that they can be mathematically described by the 2D shallow water equations. For solute transport, if the pollutant is vertically well-mixed, their dynamics may be represented by a depth-averaged advection-diffusion equation [17, 39]. Taking place in domains with irregular geometries and topographies, shallow flows are normally hydrodynamically complex and the associated solute particles are hence subject to random and complicated movement. Therefore, analytical solutions to these governing equations are generally impossible to obtain and numerical methods must be employed. Accurate numerical modelling thus provides an essential tool for water quality management, environmental impact assessment and hydraulic design [14].

It is not easy to design an accurate and efficient numerical model for solving the shallow water and advection-diffusion equations as both the flow and solute concentration may be non-smooth and contain simultaneously nonlinear bore and rarefaction waves and linear discontinuities [12]. Traditionally, a decoupled strategy is often used, i.e. the flow field is first obtained by analytical, numerical or experimental approaches and then used to drive the solute motions [3, 10, 12, 17, 31, 33]. However, Murillo et al. [34] implies that a coupling system provides a better choice in avoiding numerical instabilities in the solute concentration when it is applied to simulate complex situations where solute transport occurs in natural environmental flows with steep or even discontinuous gradients.

When discussing the Godunov-type numerical scheme for solving the 2D shallow water equations, Toro [39] suggested that the behaviour of the advection-diffusion equation for passive transport problems is identical to that of the y -direction momentum equation in the Riemann solution structure. This idea was adopted by Liang et al. [28] and they integrated the advection-diffusion equation into a pre-balanced formulation of the 2D shallow water equations to form a 4×4 hyperbolic matrix system of conservation laws. The numerical properties of the coupled system are identical to the hyperbolic shallow water equations and so most of the modern numerical techniques developed for the shallow water equations can be directly applied to solve the new 4×4 system. Liang et al. [23] solved the coupled hyperbolic conservation laws using a finite volume Godunov-type scheme on adaptive quadtree grids. This coupling strategy was also used by Murillo et al. [34] in which the integrated shallow water and solute transport equations were solved by a first-order Godunov-type scheme on unstructured grids. The scheme was later extended to include applications involving wetting and drying over complex domains [35]. A similar coupled system is also solved by Benkhaldoun et al. [4] using a non-homogeneous Riemann solver for applications involving complex bed topographies.

As mentioned before, the numerical techniques derived for the shallow water equa-