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Asymptotic Analysis of Lattice Boltzmann Outflow Treatments

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Abstract. We show the methodology and advantages of asymptotic analysis when applied to lattice Boltzmann outflow treatments. On the one hand, one can analyze outflow algorithms formulated directly in terms of the lattice Boltzmann variables, like the extrapolation method, to find the induced outflow conditions in terms of the Navier-Stokes variables. On the other hand, one can check the consistency and accuracy of lattice Boltzmann outflow treatments to given hydrodynamic outflow conditions like the Neumann or average pressure condition. As example how the gained insight can be used, we propose an improvement of the well known extrapolation method.

AMS subject classifications: 65N06, 76D05, 82C40

Key words: Outflow condition, asymptotic analysis, lattice Boltzmann method.

1 Introduction

In contrast to conventional CFD methods where the fluid velocity and pressure are primary variables, the lattice Boltzmann method recovers them as averages of the mesoscopic particle distributions in a postprocessing step (see, for example, [1, 8, 9, 12, 15, 21–23]). The advantage of very simple evolution equations for the particle distributions, however, come at the price of non-transparent relations between the desired hydrodynamic boundary conditions and the required lattice Boltzmann boundary treatments. In this article, we show that asymptotic analysis can help to clarify this relationship.

The standard lattice Boltzmann method is comprised of two phases, a collision phase and a transport phase

$$\boldsymbol{f}^c(n,\boldsymbol{j}) = \boldsymbol{f}(n,\boldsymbol{j}) - A(\boldsymbol{f}-\boldsymbol{f}^{eq})(n,\boldsymbol{j}), \tag{1.1a}$$

$$f_i(n+1,\boldsymbol{j}+\boldsymbol{c}_i) = f_i^c(n,\boldsymbol{j}). \tag{1.1b}$$

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Here, $\boldsymbol{f}(n,\boldsymbol{j})$ is the vector of particle distribution functions $f_i(n,\boldsymbol{j}) = f(n,\boldsymbol{j},\boldsymbol{c}_i)$ at the nth time level t_n and the lattice node $\boldsymbol{x_j}$ ($\boldsymbol{j} \in \mathbb{Z}^d$) with the discrete velocity $\boldsymbol{c}_i \in \{-1,0,1\}^d$ $(i = 1,2,\cdots,N)$. The particles collide locally, which is modeled with a linear operator A including BGK [23] and MRT [9, 20] approaches. The equilibrium functions f_i^{eq} recommended in [12] are adopted here,

$$f_i^{eq} = F_i(\hat{\rho},\hat{\boldsymbol{u}}), \qquad F_i(\hat{\rho},\hat{\boldsymbol{u}}) = f_i^* \Big(\hat{\rho} + 3\hat{\boldsymbol{u}}\cdot\boldsymbol{c}_i + \frac{9}{2}(\hat{\boldsymbol{u}}\cdot\boldsymbol{c}_i)^2 - \frac{3}{2}|\hat{\boldsymbol{u}}|^2\Big), \tag{1.2}$$

in which

$$\hat{\rho} = \sum_{i=1}^{N} f_i, \qquad \hat{\boldsymbol{u}} = \sum_{i=1}^{N} \boldsymbol{c}_i f_i \tag{1.3}$$

are the mass density and the average momentum of the particles based on the assumption that the fluid density slightly fluctuates around a constant $\bar{\rho}$ (here, $\bar{\rho} = 1$ without loss of generality). The constants f_i^* depend on the chosen velocity model.

The Chapman-Enskog expansion [2, 10, 11, 13] and asymptotic analysis [17, 18, 27] show that for incompressible flows governed by the Navier-Stokes equations,

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \qquad \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \nu\nabla^2\boldsymbol{u}, \qquad \boldsymbol{u}|_{t=0} = \boldsymbol{\psi}, \tag{1.4}$$

the fluid velocity $\boldsymbol{u}$ and pressure p can be extracted from the lattice Boltzmann moments $\hat{\rho}$ and $\hat{\boldsymbol{u}}$ with second order accuracy, supposing that the eigenvalues of the collision matrix A are properly related to the fluid shear viscosity ν and that initial and boundary conditions are approximated sufficiently accurate.

As far as boundary conditions are concerned, we can distinguish two basic types. (1) The Navier-Stokes problem (1.4) includes certain hydrodynamical boundary conditions (like no-slip velocity conditions, or normal stress conditions). Then, the task is to find consistent lattice Boltzmann boundary algorithms, which comes with the general difficulty that the required lattice Boltzmann boundary conditions outnumber the given hydrodynamical ones. The additional conditions have to be chosen very carefully in order to avoid conflicts on the hydrodynamical level which entail poor approximations. (2) The solution domain of (1.4) is very large or unbounded (like pipe flows or exterior flows). Then, for numerical reasons, artificial boundaries have to be introduced where no obvious physical boundary conditions are available. Again, one way to proceed is to adopt outflow conditions formulated in terms of the hydrodynamical variables and construct associated lattice Boltzmann algorithms (like the Neumann condition for the fluid velocity $\boldsymbol{u}$ [19], the do-nothing condition [19], the average pressure condition [25], or the convective condition on $\boldsymbol{u}$ [26]). Another way is to formulate reasonable outflow conditions directly for the lattice Boltzmann variables (like the extrapolation method [28], the approximation by using Grad's moments [3] and the convective condition on f_i [16, 28]). If this approach is successful, the implied conditions on the hydrodynamical level may be a valuable alternative to existing outflow treatments.