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Novel Multi-Symplectic Integrators for Nonlinear Fourth-Order Schrödinger Equation with Trapped Term

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Abstract. The multi-symplectic Runge-Kutta (MSRK) methods and multi-symplectic Fourier spectral (MSFS) methods will be employed to solve the fourth-order Schrödinger equations with trapped term. Using the idea of split-step numerical method and the MSRK methods, we devise a new kind of multi-symplectic integrators, which is called split-step multi-symplectic (SSMS) methods. The numerical experiments show that the proposed SSMS methods are more efficient than the conventional multi-symplectic integrators with respect to the the numerical accuracy and conservation perserving properties.

AMS subject classifications: 65P10, 65M06, 65M70

Key words: Schrödinger equation with trapped term, multi-symplectic scheme, Fourier spectral method, conservation law, split-step method.

1 Introduction

Considering the effect of small fourth-order dispersion term in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity, Karpman and Shagalov established the fourth-order Schrödinger equations [1–3]

$$iu_t+u_{xxxx}+\hbar'(|u|^2)u=0, \quad i=\sqrt{-1}. \tag{1.1}$$

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If an external trap potential is considered, the equation becomes the fourth-order nonlinear Schrödinger equation with a trapped term (FNSETT). In this work, we investigate the multi-symplectic integrators of the FNSETT in the form

$$iu_t+u_{xxxx}+6|u|^2u-150(\sin^2 x)u=0,\quad (x,t)\in(0,L)\times(0,T], \tag{1.2}$$
$$u(x,0)=u_0(x),\quad x\in[0,L], \tag{1.3}$$
$$u(x,t)=u(x+L,t),\quad t\in[0,T], \tag{1.4}$$

where $u_0(x)$ is a prescribed complex-valued function. The equation focuses on the most important physical effects, including dispersion, nonlinearity, and effective potential, and in the physical context, issues like Bose-Einstein condensate, nonlinear optics. The potential term $g(x)=-150\sin^2 x$ is to localize the wave around the origin. This model is a special case of the non-self-adjoint nonlinear high-order Schrödinger equation with trapped term [4–7]

$$i\frac{\partial u}{\partial t}+(-1)^m\alpha\frac{\partial^{2m}u}{\partial x^{2m}}+\frac{\partial\hbar(|u|^2)}{\partial|u|^2}u+g(x)u=0, \tag{1.5}$$

with $m=2$, $\alpha=1$, $g(x)=-150\sin^2 x$, $\hbar(|u|^2)=|u|^4$.

For the initial-boundary value problem (1.2)-(1.4), we have the following proposition.

Proposition 1.1. *The solution of the initial-boundary value problem* (1.2)-(1.4) *has at least two conserved quantities:*

1. *Charge conservation law*

$$\mathcal{Q}(t)=\int_0^L|u(x,t)|^2dx=\int_0^L|u_0(x)|^2dx=\mathcal{Q}(0); \tag{1.6}$$

2. *Energy conservation law*

$$\mathcal{E}(t)=\int_0^L[|u_{xx}|^2+3|u|^4-150(\sin^2 x)|u|^2]dx=\mathcal{E}(0). \tag{1.7}$$

Symplectic integrators have received much attention over the last decade, see, e.g., [8–12]. Recently, symplectic integrators had been generalized from Hamiltonian ODEs to Hamiltonian PDEs (HPDEs), see, e.g., [13–15]. We call this kind of numerical method multi-symplectic integrators. Many researchers are attracted by the methods for its incommensurable advantages over others for HPDEs in structure-preserving, such as in local conservation properties and in long-term numerical simulation. The method has been applied to many important physical and mathematical models, such as Schrödinger equations [16, 17], wave equations [18], Dirac equations [20], etc. It is suggested that concatenating a pair of symplectic Runge-Kutta (SRK) methods both in space and time, or concatenating an SRK method in time and Fourier spectral method in space lead to