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Efficient Dual Algorithms for Image Segmentation Using TV-Allen-Cahn Type Models

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Abstract. Variational image segmentation based on the Mumford and Shah model [31], together with implementation by the piecewise constant level-set method (PCLSM) [26], leads to fully nonlinear Total Variation (TV)-Allen-Cahn equations. The commonly-used numerical approaches usually suffer from the difficulties not only with the non-differentiability of the TV-term, but also with directly evolving the discontinuous piecewise constant-structured solutions. In this paper, we propose efficient dual algorithms to overcome these drawbacks. The use of a splitting-penalty method results in TV-Allen-Cahn type models associated with different "double-well" potentials, which allow for the implementation of the dual algorithm of Chambolle [8]. Moreover, we present a new dual algorithm based on an edge-featured penalty of the dual variable, which only requires to solve a vectorial Allen-Cahn type equation with linear $\nabla(\mathrm{div})$-diffusion rather than fully nonlinear diffusion in the Chambolle's approach. Consequently, more efficient numerical algorithms such as time-splitting method and Fast Fourier Transform (FFT) can be implemented. Various numerical tests show that two dual algorithms are much faster and more stable than the primal gradient descent approach, and the new dual algorithm is at least as efficient as the Chambolle's algorithm but is more accurate. We demonstrate that the new method also provides a viable alternative for image restoration.

AMS subject classifications: 65N22, 65N55, 74S20, 49J40

Key words: Variational models, image segmentation, TV-Allen-Cahn type equations, dual algorithms, splitting-penalty methods, FFT.

1 Introduction

Image segmentation, which aims to extract interesting objects from a given image, is one fundamental task in image processing and computer vision. Among several popular

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variational models proposed for this purpose [7, 13, 20, 31], the seminal Mumford and Shah model [31] can be formulated as: given an image f on an open bounded domain $\Omega \subseteq \mathbb{R}^2$, find a partition Ω_i of Ω and an optimal piecewise smooth approximation u of f such that u varies smoothly within each region Ω_i, but rapidly and discontinuously across the boundaries Γ of Ω_i. The partition can be done by solving the minimization problem:

$$\min_{u,\Gamma}\Big\{E_{MS}(u,\Gamma):=\mu\int_{\Omega\backslash\Gamma}|\nabla u|^2 d\boldsymbol{x}+\frac{\nu}{2}\int_{\Omega}|u-f|^2 d\boldsymbol{x}+|\Gamma|\Big\}, \tag{1.1}$$

where $|\Gamma|$ is the length of Γ, and $\mu,\nu>0$ are parameters to weight the terms in the functional. This problem involves two unknowns with different nature, and it presents a great challenge for numerical and theoretical study [2]. One may also refer to [2] for an excellent review of various attempts of approximating, relaxing or simplifying the Mumford and Shah functional as well as the numerical treatments.

Chan and Vese [13] considered a reduced version of (1.1) by assuming that the given image f consists of two phases of approximately piecewise constant intensities c_1 and c_2 on the subregions $\Omega_1=\text{inside}(\Gamma)$ and $\Omega_2=\text{outside}(\Gamma)$, respectively. Consequently, a simplification of (1.1) leads to the Chan-Vese (CV) model:

$$\min_{c_1,c_2,\Gamma}\Big\{E_{CV}(c_1,c_2,\Gamma):=\frac{\nu}{2}\Big(\int_{\Omega_1}|c_1-f|^2 d\boldsymbol{x}+\int_{\Omega_2}|c_2-f|^2 d\boldsymbol{x}\Big)+|\Gamma|\Big\}, \tag{1.2}$$

which, together with numerical implementation by the level-set method [33], has become a useful variational segmentation tool [32,39].

Lie et al. [25,26] proposed a piecewise constant level-set method (PCLSM) for image segmentation [40] and other interface problems [24,29], which enjoys some advantages over the classical level-set method. Different from the CV model, the PCLSM for (binary) Mumford and Shan image segmentation leads to a fully nonlinear PDE closely related to the Allen-Cahn equation for phase transition [1]. In contrast to the original Allen-Cahn equation with Laplace-diffusion, such a PDE involves a nonlinear TV-diffusion operator, so we term it as TV-Allen-Cahn model. Some algorithms have been developed and tested for the TV-Allen-Cahn model [11, 14, 24–26, 29, 34, 40]. However, most of them are based on directly evolving piecewise constant solutions (i.e., the primal variable), so oftentimes they suffer from numerical difficulties induced by the nonlinearity and non-differentiability the TV-term. In addition, the time evolution of piecewise-constant-structured solutions inevitably leads to a much severer restriction to the time step.

Motivated by the success of dual algorithms for image restoration [8,12,19], we propose in this paper two efficient dual algorithms to overcome the aforementioned drawbacks of the primal approaches. The key idea is to introduce an auxiliary valuable and use a splitting-penalty method, which results in a minimization problem of Rudin-Osher-Fatemi (ROF) [35] type. Accordingly, this forms the base to implement efficient dual algorithms. It is worthwhile to point out that the splitting-penalty approach has been around as early as Courant [15] in 1943, and has been applied in various context of image restoration [16, 19, 41, 43]. Indeed, it has proven to be an effective tool for convex optimization.