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Approximate Riemann Solvers and Robust High-Order Finite Volume Schemes for Multi-Dimensional Ideal MHD Equations

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Abstract. We design stable and high-order accurate finite volume schemes for the ideal MHD equations in multi-dimensions. We obtain excellent numerical stability due to some new elements in the algorithm. The schemes are based on three- and five-wave approximate Riemann solvers of the HLL-type, with the novelty that we allow a varying normal magnetic field. This is achieved by considering the semi-conservative Godunov-Powell form of the MHD equations. We show that it is important to discretize the Godunov-Powell source term in the right way, and that the HLL-type solvers naturally provide a stable upwind discretization. Second-order versions of the ENO- and WENO-type reconstructions are proposed, together with precise modifications necessary to preserve positive pressure and density. Extending the discrete source term to second order while maintaining stability requires non-standard techniques, which we present. The first- and second-order schemes are tested on a suite of numerical experiments demonstrating impressive numerical resolution as well as stability, even on very fine meshes.

AMS subject classifications: 35L65, 74S10, 65M12

Key words: Conservation laws, MHD, divergence constraint, Godunov-Powell source terms, upwinded source terms, high-order schemes.

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1 Introduction

Many interesting problems in astrophysics, solar physics and engineering involve macroscopic plasma models and are usually described by the equations of ideal magnetohydrodynamics (MHD).

1.1 Derivation of the equations

In macroscopic plasma models, the variables of interest are the mass density of the plasma ρ, the velocity field $\mathbf{u}=(u_1,u_2,u_3)^T$, the magnetic field $\mathbf{B}=(B_1,B_2,B_3)^T$, the pressure p and the total energy E. The unknowns obey the following conservation (balance) laws (see [36] for details),

1. Conservation of mass: mass of a plasma is conserved, resulting in

$$\rho_t+\mathrm{div}(\rho\mathbf{u})=0.$$

2. Faraday's law: the magnetic flux across a surface $\mathbf{S}$ bounded by a curve $\delta\mathbf{S}$ is given by Faraday's law

$$-\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS}=\int_{\delta S} E\cdot dl.$$

By using Stokes Theorem and the fact that the electric field in a co-moving frame is zero and assuming zero resistivity, Faraday's law leads to

$$\mathbf{B}_t+\mathrm{curl}(\mathbf{B}\times\mathbf{u})=-\mathbf{u}(\mathrm{div}\mathbf{B}). \tag{1.1}$$

The above equation is termed the magnetic induction equation and can also be written in the divergence form

$$\mathbf{B}_t+\mathrm{div}(\mathbf{u}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{u})=-\mathbf{u}(\mathrm{div}\mathbf{B}).$$

3. Conservation of momentum: in differential form, the conservation of momentum is

$$(\rho\mathbf{u})_t+\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}+p\mathcal{I})=\mathbf{J}\times\mathbf{B},$$

where $\mathbf{J}$ denotes the current density and $\mathcal{I}$ the 3×3 identity matrix. The Lorentz force exerted by the magnetic field is given by $\mathbf{J}\times\mathbf{B}$. Under the assumptions of ideal MHD, Ampere's law expresses the current density as

$$\mathbf{J}=\mathrm{curl}(\mathbf{B}).$$

Using standard vector identities results in the following semi-conservative form,

$$(\rho\mathbf{u})_t+\mathrm{div}\Big(\rho\mathbf{u}\otimes\mathbf{u}+\Big(p+\frac{1}{2}\mathbf{B}^2\Big)\mathcal{I}-\mathbf{B}\otimes\mathbf{B}\Big)=-\mathbf{B}(\mathrm{div}\mathbf{B}).$$