Approximate Riemann Solvers and Robust High-Order
Finite Volume Schemes for Multi-Dimensional Ideal
MHD Equations

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Abstract. We design stable and high-order accurate finite volume schemes for the
ideal MHD equations in multi-dimensions. We obtain excellent numerical stability
due to some new elements in the algorithm. The schemes are based on three- and
five-wave approximate Riemann solvers of the HLL-type, with the novelty that we
allow a varying normal magnetic field. This is achieved by considering the semi-
conservative Godunov-Powell form of the MHD equations. We show that it is im-
portant to discretize the Godunov-Powell source term in the right way, and that the
HLL-type solvers naturally provide a stable upwind discretization. Second-order ver-
sions of the ENO- and WENO-type reconstructions are proposed, together with precise
modifications necessary to preserve positive pressure and density. Extending the dis-
crete source term to second order while maintaining stability requires non-standard
techniques, which we present. The first- and second-order schemes are tested on a
suite of numerical experiments demonstrating impressive numerical resolution as well
as stability, even on very fine meshes.

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1 Introduction

Many interesting problems in astrophysics, solar physics and engineering involve macroscopic plasma models and are usually described by the equations of ideal magnetohydrodynamics (MHD).

1.1 Derivation of the equations

In macroscopic plasma models, the variables of interest are the mass density of the plasma $\rho$, the velocity field $\mathbf{u} = (u_1, u_2, u_3)^T$, the magnetic field $\mathbf{B} = (B_1, B_2, B_3)^T$, the pressure $p$ and the total energy $E$. The unknowns obey the following conservation (balance) laws (see [36] for details),

1. Conservation of mass: mass of a plasma is conserved, resulting in

$$\rho_t + \text{div}(\rho \mathbf{u}) = 0.$$  

2. Faraday’s law: the magnetic flux across a surface $S$ bounded by a curve $\delta S$ is given by Faraday’s law

$$-\frac{d}{dt} \int_S \mathbf{B} \cdot dS = \int_{\delta S} \mathbf{E} \cdot dl.$$  

By using Stokes Theorem and the fact that the electric field in a co-moving frame is zero and assuming zero resistivity, Faraday’s law leads to

$$\mathbf{B}_t + \text{curl}(\mathbf{B} \times \mathbf{u}) = -\mathbf{u}(\text{div}\mathbf{B}). \quad (1.1)$$

The above equation is termed the magnetic induction equation and can also be written in the divergence form

$$\mathbf{B}_t + \text{div}(\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) = -\mathbf{u}(\text{div}\mathbf{B}).$$

3. Conservation of momentum: in differential form, the conservation of momentum is

$$(\rho \mathbf{u})_t + \text{div}(\rho \mathbf{u} \otimes \mathbf{u} + p\mathcal{I}) = \mathbf{J} \times \mathbf{B},$$

where $\mathbf{J}$ denotes the current density and $\mathcal{I}$ the $3 \times 3$ identity matrix. The Lorentz force exerted by the magnetic field is given by $\mathbf{J} \times \mathbf{B}$. Under the assumptions of ideal MHD, Ampere’s law expresses the current density as

$$\mathbf{J} = \text{curl}(\mathbf{B}).$$

Using standard vector identities results in the following semi-conservative form,

$$(\rho \mathbf{u})_t + \text{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + \left(p + \frac{1}{2} \mathbf{B}^2\right)\mathcal{I} - \mathbf{B} \otimes \mathbf{B}\right) = -\mathbf{B}(\text{div}\mathbf{B}).$$