Commun. Comput. Phys.
doi: 10.4208/cicp.2009.09.049

Vol. **7**, No. 4, pp. 639-673
April 2010

Hyperbolic Moment Equations in Kinetic Gas Theory Based on Multi-Variate Pearson-IV-Distributions

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Received 9 March 2009; Accepted (in revised version) 27 July 2009

Communicated by Kun Xu

Available online 12 October 2009

Abstract. In this paper we develop a new closure theory for moment approximations in kinetic gas theory and derive hyperbolic moment equations for 13 fluid variables including stress and heat flux. Classical equations have either restricted hyperbolicity regions like Grad's moment equations or fail to include higher moments in a practical way like the entropy maximization approach. The new closure is based on Pearson-Type-IV distributions which reduce to Maxwellians in equilibrium, but allow anisotropies and skewness in non-equilibrium. The closure relations are essentially explicit and easy to evaluate. Hyperbolicity is shown numerically for a large range of values. Numerical solutions of Riemann problems demonstrate the capability of the new equations to handle strong non-equilibrium.

AMS subject classifications: 82B40, 35L40

Key words: Kinetic gas theory, moment approximations, hyperbolic partial differential equations.

1 Introduction

Many processes in gases are close to thermal equilibrium and can be described by fluid dynamic equations using the constitutive relations of Navier-Stokes and Fourier for stress and heat flux. In non-equilibrium processes these relations are no longer valid and the statistical description of kinetic gas theory has to be used. Moment approximations fill the gap between kinetic gas theory and fluid dynamics by deriving macroscopic partial differential equations that extend classical fluid dynamics to processes of strong non-equilibrium.

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Moment approximations have been introduced into kinetic theory by Grad [8, 9] and refined from a thermodynamic point of view by Muller and Ruggeri, see the textbook [18]. In recent years there have been considerable progress in rigorously developing a practical, stable and accurate system of moment equations for non-equilibrium gases, see [10, 26, 27, 30, 31]. In the context of computational fluid dynamics moment equations have been considered, for example in [16, 23, 29].

Moment equations are partial differential equations for an extended set of fluid variables and consist of a non-dissipative first order flux operator and a dissipative part given by relaxation and, depending on the model, also diffusion. The flux part models the free flight of the particles, while the dissipation represents collisional interaction. One of the remaining issues in the development of moment equations is the hyperbolicity of the non-dissipative flux part, that is, real-valued characteristic speeds. Hyperbolicity allows to decouple the first order flux into advection equations reflecting the free flight transport. Non-hyperbolicity renders the first order moment system mathematically ill-posed and physically useless. Unfortunately, the classical moment equations of Grad are hyperbolic only relatively close to equilibrium [4, 18, 28].

Levermore in [15] focussed on hyperbolicity of moment equations and advocated the use of the maximum entropy distribution (see also [18]). Formally, the maximum entropy distribution provides closure relations that lead to globally hyperbolic moment equations. However, due to strong non-linearity it was so far not possible to derive explicit expressions for moment systems with more than 10 fields, i.e., higher than the second moment. Unfortunately, this so-called 10-moment-system is not capable to describe heat conduction and, hence, is of little practical use for gas processes. The works [12] by Junk and [13] by Junk & Unterreiter also indicates that higher systems based on the maximum entropy distribution might face severe mathematical problems.

This paper considers a Pearson-type-IV distribution for the distribution function of the particle velocities and shows that this gives globally hyperbolic moment equations. The use of a Pearson distribution comes as an ad-hoc assumption but is justified by the fact that it reduces to the Maxwellian distribution in equilibrium and allows for skewness and anisotropies in non-equilibrium to model stress and heat flux. The resulting moment equations consider 13 fields including stress tensor and heat flux. The closure relations are almost entirely explicit and easy to evaluate. Hyperbolicity is checked by numerical evaluation over a wide range of variable states. Additionally, Riemann problems are computed to demonstrate the ability of the new system to cope with strong non-equilibria.

Hyperbolicity is part of a larger group of properties desirable for a system of moment equations. This group contains for example Galilei-invariance, hyperbolicity, the existence of an entropy, and a positive realizable distribution function. The importance and relation of these properties is not clear and partly depends on the philosophy of the model. Ideally, a moment system would exhibit all these properties, but in general, it seems that they partly exclude each other. In this paper all systems of equations will be Galilei-invariant and hyperbolicity is the ultimate goal. To achieve this, we may com-