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A Robust WENO Type Finite Volume Solver for Steady Euler Equations on Unstructured Grids

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To the memory of David Gottlieb

Abstract. A recent work of Li et al. [Numer. Math. Theor. Meth. Appl., 1(2008), pp. 92-112] proposed a finite volume solver to solve 2D steady Euler equations. Although the Venkatakrishnan limiter is used to prevent the non-physical oscillations nearby the shock region, the overshoot or undershoot phenomenon can still be observed. Moreover, the numerical accuracy is degraded by using Venkatakrishnan limiter. To fix the problems, in this paper the WENO type reconstruction is employed to gain both the accurate approximations in smooth region and non-oscillatory sharp profiles near the shock discontinuity. The numerical experiments will demonstrate the efficiency and robustness of the proposed numerical strategy.

AMS subject classifications: 65N08, 65N22, 65N55, 76G25

Key words: Steady Euler equations, finite volume method, WENO reconstruction, geometrical multigrid, Block LU-SGS.

1 Introduction

Recently, Li et al. [17] proposed a finite volume solver for 2D steady Euler equations. In the algorithm, the Newton-iteration method is adopted to linearize the Euler equation, and in each Newton-iteration the multigrid method with block lower-upper symmetric Gauss-Seidel (LU-SGS) iteration as its smoother is used to solve the linearized system. In

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the reconstruction step, the linear reconstruction is employed to describe the variation of solutions in each cell. To avoid the non-physical oscillations, the Venkatakrishnan limiter (VL) is adopted to constrain gradients during the reconstruction process.

The limiting strategy is very important for simulations with the finite volume method. A useful limiter function should be able to remove the non-physical oscillations nearby the shock profiles and can also preserve the numerical accuracy in the smooth regions. Moreover, the limiter function should not affect the convergence to the steady state. So far, many useful limiting strategies for the structured mesh have been proposed, including the total variation diminishing (TVD) limiter [8, 9], the slope limiters like minmod limiter, the superbee limiter, the MC limiter and van Leer limiter (all these limiters can be found in [15] and references therein). However, since the fixed stencil is used to approximate the variation of solutions, the numerical accuracy is always degraded when the above limiter functions are used. To preserve the numerical accuracy, the essentially non-oscillatory (ENO) method was introduced [10, 11]. In order to reconstruct the approximate polynomial of solutions on each cell, the ENO methods test different neighboring stencils so that the locally smoothest stencil is selected eventually. By selecting a convex combination of results obtained from all possible stencils, the weighted essentially non-oscillatory (WENO) methods were proposed, see, e.g., [14,18,27]. These limiter functions yield satisfactory numerical results on structured meshes.

On unstructured meshes, one of the classical ways to obtain high resolution results is to use the *k-exact* reconstruction [1, 24] together with a slope limiter. For example, for the linear case, it is assumed that the solution is piecewise linearly distributed over the cell. Such linear approximation is determined by solving a least square system based on cell averages of the cell and its neighbours. After that, certain slope limiter is used to guarantee the monotonicity of solutions. On the unstructured meshes, the first implementation of a limiter function was presented by Barth and Jespersen [3]. The Barth and Jespersen limiter is used to enforce a monotone solution. However, their method is rather dissipative which leads to smear discontinuities. Furthermore, the limiter may be active in smooth flow regions due to the numerical noise, which causes difficulties for steady state convergence [6]. To improve the differentiability of the limiter in [3], the VL was proposed in [29] and has been widely used. Similar to the structured mesh case, the theoretically predicted accuracy also can not be guaranteed with the fixed stencil when these limiters are used. Moreover, since the VL does not preserve strict monotonicity, slight oscillations can be observed near shock discontinuities. To further improve the quality of numerical solutions for unstructured meshes, the ENO/WENO type reconstructions may be considered due to their good performance on the structured meshes.

Many works have been done for using WENO methods as the limiting strategy. In [26], a Hermite WENO scheme is proposed for the one-dimensional problems, and it is used as limiters for Runge-Kutta discontinuous Galerkin method. Then Luo et al. [21] give an implementation of Hermite WENO-based limiter on the unstructured grids. In [22, 33], the WENO type methods are also adopted as the limiting strategy to the discontinuous Galerkin method. With the help of the WENO method, all these methods demonstrate