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A Weighted Runge-Kutta Method with Weak Numerical Dispersion for Solving Wave Equations

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Abstract. In this paper, we propose a weighted Runge-Kutta (WRK) method to solve the 2D acoustic and elastic wave equations. This method successfully suppresses the numerical dispersion resulted from discretizing the wave equations. In this method, the partial differential wave equation is first transformed into a system of ordinary differential equations (ODEs), then a third-order Runge-Kutta method is proposed to solve the ODEs. Like the conventional third-order RK scheme, this new method includes three stages. By introducing a weight to estimate the displacement and its gradients in every stage, we obtain a weighted RK (WRK) method. In this paper, we investigate the theoretical properties of the WRK method, including the stability criteria, numerical error, and the numerical dispersion in solving the 1D and 2D scalar wave equations. We also compare it against other methods such as the high-order compact or so-called Lax-Wendroff correction (LWC) and the staggered-grid schemes. To validate the efficiency and accuracy of the method, we simulate wave fields in the 2D homogeneous transversely isotropic and heterogeneous isotropic media. We conclude that the WRK method can effectively suppress numerical dispersions and source noises caused in using coarse grids and can further improve the original RK method in terms of the numerical dispersion and stability condition.

AMS subject classifications: 65M06, 65M12, 86-08, 86A15

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1 Introduction

Finite difference (FD) is the most widely used numerical scheme in solving the wave equation for wave propagation in seismology. The two most widely used "families" of

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FD methods are the compact schemes or so-called LWC methods and the staggered-grid schemes. To reduce the computation and memory usage, the high-order compact FD scheme was proposed [1] and widely applied (e.g., [6, 10, 19]). The staggered-grid FD scheme was first developed by Madariaga [12] to model an expanding circular crack in an elastic space. Virieux [17, 18] developed a velocity-stress staggered-grid FD scheme to simulate wave propagating in heterogeneous media. To improve the accuracy and increase the efficiency, Levander [11] developed a fourth-order staggered-grid scheme, and this FD method was later extended to different cases (e.g., [3–5, 13, 16]).

A main reason that the above two kinds FD schemes are so popular is that they can reduce the numerical dispersion resulted from the discretization of wave equations. However, numerical dispersion may still exist when too few samples per wavelength are used [15, 22]. The nearly analytic discrete method (NADM) [25] proposed recently and its improved version (INADM) [21] are much superior with regard to suppressing the numerical dispersion. These methods, based on the truncated Taylor expansion and the local interpolation compensation for the truncated Taylor series, use the wave displacement-, the velocity- and their gradient-fields to reconstruct the wave displacement fields. Thus they can effectively suppress the numerical dispersion, as compared to the compact FD methods [21, 24, 25].

In this paper, we develop an alternate weighted RK method to further suppress the numerical dispersion. In this method, we first use the high-order interpolation approximations to approximate the high-order spatial derivatives and convert the wave equation into a system of ordinary differential equations (ODEs), and then we solve the converted ODEs by using the explicit third-order RK method [7, 14] that is similar to the RK method developed by Yang et al. [20]. Due to the multistage property of the RK method, we introduce a weight to evaluate the displacement and its gradients in every stage, which results in the method called the weighted Runge-Kutta (WRK) method. Since the method not only uses the wave displacement and the particle velocity but also their gradient-fields to reconstruct the wave displacement fields, the WRK method can suppress effectively the numerical dispersion like our previous RK method [20]. But the weight introduced here makes the WRK method much more effective in suppressing the numerical dispersion and in enhancing the stability condition by comparing with the original third-order RK method. However, it should be mentioned that the weight may reduce the accuracy in time.

2 Theory of the WRK method

Third-order Runge-Kutta method

Consider the following differential equation

$$\frac{du}{dt} = L(u). \tag{2.1}$$