Commun. Comput. Phys.
doi: 10.4208/cicp.071109.080710a

Vol. **9**, No. 4, pp. 937-958
April 2011

Effect of Element Distortion on the Numerical Dispersion of Spectral Element Methods

S. P. Oliveira[1] and G. Seriani[2,*]

[1] *Departamento de Matemática, Universidade Federal do Paraná, Curitiba-PR, 81531-980, Brazil.*
[2] *Istituto Nazionale di Oceanografia e di Geofisica Sperimentale, Borgo Grotta Gigante, 42/c, Sgonico (TS), 34010, Italy.*

Received 7 November 2009; Accepted (in revised version) 8 July 2010

Communicated by Jan S. Hesthaven

Available online 2 November 2010

Abstract. Spectral element methods are well established in the field of wave propagation, in particular because they inherit the flexibility of finite element methods and have low numerical dispersion error. The latter is experimentally acknowledged, but has been theoretically shown only in limited cases, such as Cartesian meshes. It is well known that a finite element mesh can contain distorted elements that generate numerical errors for very large distortions. In the present work, we study the effect of element distortion on the numerical dispersion error and determine the distortion range in which an accurate solution is obtained for a given error tolerance. We also discuss a double-grid calculation of the spectral element matrices that preserves accuracy in deformed geometries.

AMS subject classifications: 65C20, 65M60, 65M70, 65M15, 86A15, 65Z05

Key words: Spectral element method, dispersion analysis, slivering.

1 Introduction

Spectral element methods are high-order finite element methods that use collocation points derived from orthogonal polynomials with the goal of enhancing approximation properties, in the spirit of spectral methods [3]. Spectral elements have been used in fluid mechanics [12, 17, 26, 29] and wave propagation [5, 13, 16, 28, 36]. Both triangular (tetrahedral) and quadrilateral (hexahedral) elements have been developed with various collocation point schemes, which for quadrilateral (hexahedral) element may take the simple form of a cartesian product of either Gauss-Lobatto-Legendre (GLL, [4, 5, 12]) or Gauss-Lobatto-Chebyshev (GLC, [9, 18, 37, 39]) nodes.

*Corresponding author. *Email addresses:* `saulopo@ufpr.br` (S. P. Oliveira), `gseriani@inogs.it` (G. Seriani)

These methods are particularly popular in computational seismology because they are able to simulate long-wave propagation events with low numerical dispersion, when compared with finite difference or standard finite element methods. Quadrilateral (hexahedral in 3D) meshes are the standard choice in computational seismology [5] because they naturally inherit the best approximation properties from the optimality of the collocation points and the algorithmic structure from the 1D case. Mercerat et al. [23] applied recent triangular spectral element techniques to elastic wave propagation problems and noted that quadrangular elements are more efficient in these problems, although high-order methods with triangular and tetrahedral meshes have shown to be competitive in other fields [13]. However, mesh design is a major bottleneck in this field, and it is necessary to seek a trade-off between honouring geological features and keeping the deformation of the mesh elements small enough for the desired accuracy [5]. In fact, realistic complex geological model automatic mesh generators (e.g., [18, 27]) may produce millions of elements (or even more) that cannot be visually verified and some of them can be very distorted (slivers). Knowing in advance the maximum allowed error for a simulation, a maximum element distortion can then be imposed to the mesh generator for the desired accuracy.

The effect of mesh deformation on the accuracy of finite elements has long been recognized. Although optimal error estimates may be preserved, the constant present on the error estimate becomes progressively larger as the distortion increases [15]. Maday and Rønquist [20] analyzed GLL spectral elements on deformed geometries and find an estimate whose constant is greater than $|1/J|$, where J is the determinant of the Jacobian of the transformation. Note that J may be zero for cases where quadrilaterals collapse into triangles by coalescing two nodes [15].

Some of the known sources of error in distorted meshes are the ill conditioning generated when corner angles are nearly 180° [2] and the fact that the computation of stiffness matrices may lead to rational integrands [40]. While the former relies on mesh design, the latter can be controlled by numerical or analytical integration techniques (see current approaches to this issue in [40]).

Some authors have briefly addressed the accuracy of spectral elements on distorted meshes. Melenk et al. [22] advocated the need of over-integration to account for mesh distortion for Legendre spectral elements, while in [7, 20] this method was found to perform well with a standard set of integration points, even for large distortions.

One of the most significant measures of the accuracy of wave propagation modeling is numerical dispersion. Dispersion analysis is an essential tool for setting up discretization parameters for an efficient numerical simulation of wave propagation with a prescribed level of accuracy. The dispersion properties of low-order and high-order finite elements are well understood on Cartesian grids [1, 10, 32], but few authors have considered deformed grids [7, 38].

The purpose of this paper is to describe the dependency of accuracy on mesh distortion in a systematic manner, providing a better understanding of spectral element methods on complex geometries, as well as unresolved issues such as the need of over-