A Well-Conditioned Hierarchical Basis for Triangular *H*(**curl**)-Conforming Elements

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To the memory of David Gottlieb

Abstract. We construct a well-conditioned hierarchical basis for triangular *H*(**curl**)-conforming elements with selected orthogonality. The basis functions are grouped into edge and interior functions, and the later is further grouped into normal and bubble functions. In our construction, the trace of the edge shape functions are orthonormal on the associated edge. The interior normal functions, which are perpendicular to an edge, and the bubble functions are both orthonormal among themselves over the reference element. The construction is made possible with classic orthogonal polynomials, viz., Legendre and Jacobi polynomials. For both the mass matrix and the quasi-stiffness matrix, better conditioning of the new basis is shown by a comparison with the basis previously proposed by Ainsworth and Coyle [Comput. Methods. Appl. Mech. Engrg., 190 (2001), 6709-6733].

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1 Introduction

The Nédélec elements [17] are the natural choices when problems in electromagnetism are solved by finite element methods. Hierarchical bases are more convenient to use when a *p*-refinement technique is applied with the finite element methods [6,7]. Webb [27] constructed hierarchical vector bases of arbitrary order for triangular and tetrahedral finite elements. It was shown [11] that the basis functions in [27] indeed span the true Nédélec space [17]. A basis in terms of affine coordinates was also given [11]. Inspired by the foundational work [17] and following Webb [27], many researchers had constructed

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various hierarchical bases for several common elements in 2D and 3D [1,3–5,14,15,18,22, 25]. Meanwhile, using the perspective of differential forms, Hiptmair [12] laid a general framework for canonical construction of $H(\mathbf{curl})$- and $H(\mathbf{div})$-conforming finite elements. For further details, the reader is referred to the works [13,19–21] and the monograph [8].

One problem with hierarchical bases is the ill-conditioning of the finite element discretization matrices for the Maxwell's equations when higher-order bases are applied [2, 27,29]. For a hierarchical basis to be useful, the issue of ill-conditioning has to be resolved. Using Gram-Schmidt orthogonalization procedure, Webb [27] gave the explicit formulas of the basis functions up to third order for triangular and tetrahedral elements. Following the same line of development [27], i.e., decomposing the basis functions into rotational and irrotational groups, Sun and collaborators [25] investigated the conditioning issue more carefully and also gave the basis functions up to the third order. Ainsworth and Coyle [3] studied both the dispersive and conditioning issues for the hierarchical basis on hybrid quadrilateral/triangular meshes. With the aid of Jacobi polynomials, the interior bubble functions are orthogonal over the equilateral reference triangle [3]. With this partial orthogonality it was shown that the condition numbers of both the mass matrix and the stiffness matrix could be reduced significantly [3]. Using Legendre polynomials, Jørgensen et al. constructed a near-orthogonal basis for the quadrilaterals and indicated that the same procedure could be applied for the triangles with the help of collapsed coordinate system [16]. More recently, Schöberl and Zaglmayr [22] created bases for high-order Nédélec elements with the property of local complete sequence to partially address the ill-conditioning issue. The key component in their construction [22] is to use (i) the gradients of scalar basis functions and, (ii) scaled and integrated Legendre polynomials. However, the ill-conditioning issue was pronounced with higher-order approximation and moderate growth of the condition number was reported [22]. A new hierarchical basis with uncommon orthogonality properties was constructed by Ingelström [14] for tetrahedral meshes where higher-order basis functions vanished if they were projected onto the relatively lower-order $H(\mathbf{curl})$-conforming spaces [14]. It was shown [14] that such a basis was well suited for use with multi-level solvers. Recently, using the orthogonalization procedure by Shreshevskii [24] and conforming to the Nédélec [17] condition, Abdul-Rahman and Kasper proposed a new hierarchical basis for the tetrahedral element [1].

The Gram-Schmidt scheme used by Webb [27] or the orthogonalization method applied by Abdul-Rahman and Kasper [1] involves a linear system of equations to be solved, and the coefficients associated with the basis functions in general cannot be expressed in closed forms. The focus of the current work is to construct a well-conditioned hierarchical basis for the triangular $H(\mathbf{curl})$-conforming elements without using the Gram-Schmidt orthogonalization. This is accomplished by using integrated Legendre polynomials for higher-order edge functions and Jacobi polynomials for interior functions. The basis functions of any approximation order are given explicitly in closed form. Our work is based upon the studies by Ainsworth and Coyle [3], and by Schöberl and Zaglmayr [22], and motivated by the study of orthogonal polynomials of several variables [9].