

Constraint Preserving Schemes Using Potential-Based Fluxes I. Multidimensional Transport Equations

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To the memory of David Gottlieb

Abstract. We consider constraint preserving multidimensional evolution equations. A prototypical example is provided by the magnetic induction equation of plasma physics. The constraint of interest is the divergence of the magnetic field. We design finite volume schemes which approximate these equations in a stable manner and preserve a discrete version of the constraint. The schemes are based on reformulating standard edge centered finite volume fluxes in terms of vertex centered potentials. The *potential-based* approach provides a general framework for faithful discretizations of constraint transport and we apply it to both divergence preserving as well as curl preserving equations. We present benchmark numerical tests which confirm that our potential-based schemes achieve high resolution, while being constraint preserving.

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1 Introduction

We are concerned with evolution equations of the form

$$\mathbf{u}_t + L(\partial_x, \mathbf{f}(x, t, \mathbf{u})) = 0, \quad \forall (x, t) \in \mathbb{R}^n \times \mathbb{R}_+, \quad (1.1)$$

where $\mathbf{u}(x, t) : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}^m$ is the unknown, $\mathbf{f} : X \mapsto X$ is a nonlinear flux function and $L : X \mapsto Y$ is a differential operator acting on the Sobolev space X . We assume there exists

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another differential operator $M: Y \mapsto Z$, such that $ML(\mathbf{f}(\cdot, \cdot, \mathbf{v})) \equiv 0$ for all $\mathbf{v} \in X$. Applying the operator M to both sides of (1.1), we obtain

$$(M\mathbf{u})_t \equiv 0. \quad (1.2)$$

Hence, solutions of (1.1) satisfy an additional constraint which enforces them to lie on a sub-manifold of the space X .

The above framework is generic to a large class of evolution equations involving *intrinsic constraints*. We mention three prototype examples. As a first example, consider the curl advection

$$\mathbf{u}_t + \text{curl}(\mathbf{f}(x, t, \mathbf{u})) = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+. \quad (1.3)$$

This equation is an example for (1.1) and (1.2), with the differential operators $L = \text{curl}$ and $M = \text{div}$. Hence, solutions of (1.3) satisfy the additional divergence constraint

$$\text{div}(\mathbf{u})_t = 0. \quad (1.4)$$

A specific example for (1.3) is the magnetic induction equation of plasma physics. Under the assumptions of zero resistivity, the magnetic field \mathbf{u} , evolving under the influence of a given velocity \mathbf{v} , satisfies the following form of the Maxwell's equations [23]

$$\mathbf{u}_t + \text{curl}(\mathbf{u} \times \mathbf{v}) = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+. \quad (1.5)$$

The fact that magnetic monopoles have not been observed in nature implies that

$$\text{div}(\mathbf{u}(x, 0)) \equiv 0. \quad (1.6)$$

As a consequence of the divergence constraint (1.4), the solutions of (1.5) remain divergence free. The magnetic induction equation (1.5) is a sub-model for the equations of ideal Magnetohydrodynamics (MHD) [11].

Adding magnetic resistivity to the model leads to the viscous magnetic induction equations

$$\mathbf{u}_t + \text{curl}(\mathbf{u} \times \mathbf{v}) = -\sigma(\text{curl}(\text{curl}\mathbf{u})), \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+. \quad (1.7)$$

The parameter σ is the resistivity co-efficient of the medium. Solutions of (1.7) also satisfy the divergence constraint (1.4).

A second example for (1.1) and (1.2) is the grad advection

$$\mathbf{w}_t + \text{grad}(\mathbf{f}(x, t, \mathbf{w})) = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}_+. \quad (1.8)$$

The differential operators of interest are $L = \text{grad}$ and $M = \text{curl}$ and solutions of (1.8) satisfy the additional constraint

$$\text{curl}(\mathbf{w})_t = 0.$$