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An Energy Regularization for Cauchy Problems of Laplace Equation in Annulus Domain

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Abstract. Detecting corrosion by electrical field can be modeled by a Cauchy problem of Laplace equation in annulus domain under the assumption that the thickness of the pipe is relatively small compared with the radius of the pipe. The interior surface of the pipe is inaccessible and the nondestructive detection is solely based on measurements from the outer layer. The Cauchy problem for an elliptic equation is a typical ill-posed problem whose solution does not depend continuously on the boundary data. In this work, we assume that the measurements are available on the whole outer boundary on an annulus domain. By imposing reasonable assumptions, the theoretical goal here is to derive the stabilities of the Cauchy solutions and an energy regularization method. Relationship between the proposed energy regularization method and the Tikhonov regularization with Morozov principle is also given. A novel numerical algorithm is proposed and numerical examples are given.

AMS subject classifications: 65N12, 65N15, 65N21

Key words: Inverse problem, stability, error bounds, Tikhonov regularization, method of fundamental solution.

1 Introduction

To provide a safe and reliable mode of energy transport and improve general awareness of the benefits of the pipeline industry, corrosion prevention, preventing outside damage are the high priority tasks. Nevertheless, the recent promotion of high-pressure pipelines

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by different *Australian Pipeline Industry Association* makes the problems of corrosion detection equally significant. According to the *Association of Oil pipe Lines* (AOPL), the most recent *Crack Detection Tools* includes ultrasonic crack detection, magnetic flux leakage, and elastic wave tool. "*Some of these technologies are fairly new and still developing*," quoted from the AOPL's Pipeline Industry Facts. The wide variety of defects in pipelines can be classified into three major groups: defects due to corrosion, defects generated by mechanical damage, and cracks created by stresses in the pipe wall. Defects of these three groups are mathematically different; each requires specific techniques and models.

If we focus on defects due to corrosion, the *inverse problem* considered is to nondestructively determine information about the corrosion that occurs on the interior surface of the pipeline. The only accessible data are the electrostatic measurements on the exterior surface of the pipeline. We assume that the thickness of the pipeline is comparatively small with respect to its diameter. This includes the crude oil pipeline: for example, the 2005 El Sharara-Mellitah Onshore Pipeline project of the AGIP Oil Company engaged a 400km of pipelines of 762mm overall diameter with 11.30mm thickness.

Inglese [1] modeled the problem of determining quantitative information about corrosion by Laplace's equation

$$\triangle u = 0, \qquad x \in \Omega, \tag{1.1}$$

with unknown (interior) boundary conditions. As the thickness of the coating goes to zero, Buttazzo and Kohn [2] observed the arising of mixed boundary conditions. Based on the Faraday's law, corrosion or mass loss is proportional to the normal current flux. The main focuses of this paper are the stability and the numerical algorithm for this inverse problem. Surface with corrosion is usually rough on which a thin coating effect applies. Let Γ_{out} and Γ_{in} be the outer and inner boundary of the annulus domain Ω, respectively. After linearization, one gets

$$u_\nu(x) + \gamma(x) u(x) = 0, \qquad x \in \Gamma_{in}, \tag{1.2}$$

where u_ν is the outer normal derivative of u on the boundary. Using the potential model introduced by Ingless and Santosa [3], defects due to corrosion can be described by mixed boundary conditions. In (1.2), $\gamma(x)$ is the coefficient of energy exchange and $\gamma(x) \geq 0$ indicates corrosion damage. The inverse problem here is to find $\gamma(x)$ given knowledge of

$$u(x) = f(x) \quad \text{and} \quad u_\nu(x) = g(x), \qquad x \in \Gamma_{out}. \tag{1.3}$$

Derived from (1.1) and (1.2), the conservation of charge forces

$$\int_{\Gamma_{out}} g - \int_{\Gamma_{in}} \gamma u = 0, \tag{1.4}$$

indicates that the flux through Γ_{out} is not zero. Protter and Weinberger [4] showed that, without loss of generality, choosing $f \geq 0$ ensure $u > 0$ in $\overline{\Omega}$ if both γ and f have positive Lebesgue measure. The exchange coefficient could be computed by first solving the