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Numerical Simulation of Time-Harmonic Waves in Inhomogeneous Media using Compact High Order Schemes

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Dedicated to the memory of our dear friend, David Gottlieb

Abstract. In many problems, one wishes to solve the Helmholtz equation with variable coefficients within the Laplacian-like term and use a high order accurate method (e.g., fourth order accurate) to alleviate the points-per-wavelength constraint by reducing the dispersion errors. The variation of coefficients in the equation may be due to an inhomogeneous medium and/or non-Cartesian coordinates. This renders existing fourth order finite difference methods inapplicable. We develop a new compact scheme that is provably fourth order accurate even for these problems. We present numerical results that corroborate the fourth order convergence rate for several model problems.

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Key words: Helmholtz equation, variable coefficients, high order accuracy, compact finite differences.

1 Introduction

In many problems in computational electrodynamics one considers media with variable properties. Our goal is to obtain high order schemes for the corresponding wave propagation problems. Consider the two dimensional (TE_z) Maxwell equations in frequency

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space:

$$-i\omega\mu H_z = \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x},$$
$$-i\omega E_x = \frac{1}{\varepsilon}\frac{\partial H_z}{\partial y}, \qquad i\omega E_y = \frac{1}{\varepsilon}\frac{\partial H_z}{\partial x}.$$

Combining those into a single second order equation, we have:

$$0 = \frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial H_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\varepsilon}\frac{\partial H_z}{\partial y}\right) + \mu\omega^2 H_z.$$

More generally, we consider the following 2D variable coefficient Helmholtz equation:

$$\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b(x,y)\frac{\partial u}{\partial y}\right) + k^2(x,y)u(x,y) = 0. \tag{1.1}$$

We emphasize that in many cases it is both easier and cheaper to solve a single second order equation, such as Eq. (1.1), rather than the underlying system of first order equations, see, e.g., [16, 17, 20]. We also stress that the coefficients of Eq. (1.1) vary inside the derivatives. Hence, a straightforward Padé approximation will not work. Because of the pollution effect [3, 6], second order accurate schemes are very inefficient, especially for high frequencies. Our aim is to construct a fourth order accurate finite difference scheme, which would have a compact 9 point stencil in two dimensions (and 27 points in three dimensions). Note that having a small stencil or, in other words, having the same (second) order of the difference equation as that of the differential equation, yet with high order accurate approximation, is convenient, as it considerably simplifies setting the boundary conditions [5, 7] and also leads to a narrower bandwidth of the resulting matrix.

Nehrbass, Jevtic, and Lee studied ways of reducing the phase error [19]. They used a 5 point stencil and replaced the weight of the center node using a Bessel function. Harari and Turkel [15] constructed a fourth order approximation for the Helmholtz equation subject to Dirichlet boundary conditions. The method was based on Padé expansions. It was extended by Singer and Turkel [22] to Neumann boundary conditions. They also introduced an approach referred to as equation based. In this approach, one finds the truncation error of a classical second order method and then uses the Helmholtz equation and its derivatives to eliminate this truncation error to the next order. In both cases, the coefficients a and b in (1.1) were required to be constant, though k could be a smooth function of x and y. A different approach was used by Caruthers, Steinhoff, and Engels [8], who based a difference approximation on Bessel functions. This approach requires that all the coefficients be constant. Under this assumption one can even construct sixth order accurate approximations, see, e.g., [18, 23, 26].

Besides the variation of physical properties of the medium leading to Eq. (1.1), the coefficients of a differential equation may vary because the equation is expressed in non-Cartesian coordinates. In the recent paper [7], we have constructed a fourth order accurate compact finite difference scheme for the Helmholtz equation in polar coordinates. In