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CONVERGENCE AND OPTIMALITY OF ADAPTIVE MIXED METHODS FOR POISSON'S EQUATION IN THE FEEC FRAMEWORK*

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Abstract

Finite Element Exterior Calculus (FEEC) was developed by Arnold, Falk, Winther and others over the last decade to exploit the observation that mixed variational problems can be posed on a Hilbert complex, and Galerkin-type mixed methods can then be obtained by solving finite-dimensional subcomplex problems. Chen, Holst, and Xu (Math. Comp. 78 (2009) 35–53) established convergence and optimality of an adaptive mixed finite element method using Raviart–Thomas or Brezzi–Douglas–Marini elements for Poisson's equation on contractible domains in $\mathbb{R}^2$, which can be viewed as a boundary problem on the de Rham complex. Recently Demlow and Hirani (Found. Math. Comput. 14 (2014) 1337–1371) developed fundamental tools for a posteriori analysis on the de Rham complex. In this paper, we use tools in FEEC to construct convergence and complexity results on domains with general topology and spatial dimension. In particular, we construct a reliable and efficient error estimator and a sharper quasi-orthogonality result using a novel technique. Without marking for data oscillation, our adaptive method is a contraction with respect to a total error incorporating the error estimator and data oscillation.

Mathematics subject classification: 65N12, 65N15, 65N30, 65N50.
Key words: Finite Element Exterior Calculus, Adaptive finite element methods, A posteriori error estimates, Convergence, Quasi-optimality.

1. Introduction

An idea that has had a major influence on the development of numerical methods for PDE applications is that of mixed finite elements, whose early success in areas such as computational electromagnetics was later found to have surprising connections with the calculus of exterior differential forms, including de Rham cohomology and Hodge theory [9, 19, 30, 31]. A core idea underlying these developments is the Helmholtz-Hodge orthogonal decomposition of an arbitrary vector field $f \in (L^2(\Omega))^3$ into curl-free, divergence-free, and harmonic functions:

$$f = \nabla p + \nabla \times q + h,$$

where $p \in H_0^1(\Omega), q \in H(\text{curl}, \Omega)$, and h is harmonic (divergence- and curl-free). The mixed formulation is explicitly computing the decomposition for $h = 0$, and finite element methods based on mixed formulations exploit this. There is a connection between this decomposition

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and de Rham cohomology; the space of harmonic forms is isomorphic to the first de Rham cohomology of the domain Ω, with the number of holes in Ω giving the first Betti number, and creating obstacles to well-posed formulations of elliptic problems. A natural question is then: What is an appropriate mathematical framework for understanding this abstractly, that will allow for a methodical construction of "good" finite element methods for these types of problems? The answer turns out to be the theory of Hilbert Complexes. Hilbert complexes were originally studied in [11] as a way to generalize certain properties of elliptic complexes, particularly the Hodge decomposition and other aspects of Hodge theory. The Finite Element Exterior Calculus (FEEC) [3, 4] was developed to exploit this abstraction. A key insight was that from a functional-analytic point of view, a mixed variational problem can be posed on a Hilbert complex: a differential complex of Hilbert spaces, in the sense of [11]. Galerkin-type mixed methods are then obtained by solving the variational problem on a finite-dimensional subcomplex. Stability and consistency of the resulting method, often shown using complex and case-specific arguments, are reduced by the framework to simply establishing existence of operators with certain properties that connect the Hilbert complex with its subcomplex, essentially giving a "recipe" for the development of provably well-behaved methods.

Due to the pioneering work of Babuška and Rheinboldt [5], adaptive finite element methods (AFEM) based on a posteriori error estimators have become standard tools in solving PDE problems arising in science and engineering (cf. [1, 34, 38]). A standard adaptive algorithm has the general iterative structure:

$$\mathsf{Solve} \longrightarrow \mathsf{Estimate} \longrightarrow \mathsf{Mark} \longrightarrow \mathsf{Refine}, \tag{1.1}$$

where Solve computes the discrete solution u_ℓ in a subspace $X_\ell \subset X$; Estimate computes certain error estimators based on u_ℓ, which are reliable and efficient in the sense that they are good approximation of the true error $u - u_\ell$ in the energy norm; Mark applies certain marking strategies based on the estimators; and finally, Refine divides each marked element and completes the mesh to obtain a new partition, and subsequently an enriched subspace $X_{\ell+1}$. The fundamental problem with the adaptive procedure (1.1) is guaranteeing convergence of the solution sequence. The first convergence result for (1.1) was obtained by Babuška and Vogelius [6] for linear elliptic problems in one space dimension. The multi-dimensional case was open until Dörfler [18] proved convergence of (1.1) for Poisson's equation by using the so called Dörfler marking, under the assumption that the initial mesh was fine enough to resolve the influence of data oscillation. This result was improved by Morin, Nochetto, and Siebert [28], in which the convergence was proved without conditions on the initial mesh, but requiring the so-called interior node property, together with an additional marking step driven by data oscillation. It was shown by Binev, Dahmen and DeVore [8] for the first time that AFEM for Poisson's equation in the plane has optimal computational complexity by using a special coarsening step. This result was improved by Stevenson [36] by showing the optimal complexity in general spatial dimension without a coarsening step. These error reduction and optimal complexity results were improved in several aspects in [12]. In their analysis, the artificial assumptions of interior node and extra marking due to data oscillation were removed, and the convergence result is applicable to general linear elliptic equations. The main ingredients of this new convergence analysis are the global upper bound on the error given by the a posteriori estimator, orthogonality (or possibly only quasi-orthogonality) of the underlying bilinear form arising from the linear problem, and a type of error indicator reduction produced by each step of AFEM. In another direction, Morin, Siebert, and Veeser [29] gave a plain convergence result