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MIXED FINITE ELEMENT METHODS FOR FRACTIONAL NAVIER-STOKES EQUATIONS*

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Abstract

This paper gives the detailed numerical analysis of mixed finite element method for fractional Navier-Stokes equations. The proposed method is based on the mixed finite element method in space and a finite difference scheme in time. The stability analyses of semi-discretization scheme and fully discrete scheme are discussed in detail. Furthermore, We give the convergence analysis for both semidiscrete and fully discrete schemes and then prove that the numerical solution converges the exact one with order $O(h^2 + k)$, where h and k respectively denote the space step size and the time step size. Finally, numerical examples are presented to demonstrate the effectiveness of our numerical methods.

Mathematics subject classification: 60N15, 65M60, 60N30, 75D05.
Key words: Time-fractional Navier-Stokes equations, Finite element method, Error estimates; Strong convergence.

1. Introduction

The purpose of the present paper is to study the error estimates of the mixed finite element method for the incompressible fractional Navier-Stokes equations

$$\begin{cases} u_t + \mathscr{B}^\alpha \mathscr{L} u + u \cdot \nabla u + \nabla p = f, & \text{in } \ \Omega \times [0,T], \\ \nabla \cdot u = 0, & \text{in } \ \Omega \times [0,T], \\ u(x,0) = u_0, & \text{in } \ \Omega, \\ u = 0, & \text{on } \ \partial\Omega \times [0,T], \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded and connected polygonal domain, u represents the velocity field, p is the associated pressure, u_0 is the initial velocity and f is an external force, $\mathscr{L}u = -\nu \triangle u$ ($\nu > 0$ is the viscosity coefficient), $\mathscr{B}^\alpha :=^R D_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative in time defined by: for $0 < \alpha < 1$,

$$\mathscr{B}^\alpha \varphi(t) := \frac{\partial}{\partial t} \mathscr{I}^\alpha \varphi(t) := \frac{\partial}{\partial t} \int_0^t \omega_\alpha(t-s)\varphi(s) ds \ \text{ with } \ \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tag{1.2}$$

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with $\mathscr{I}^\alpha$ being the temporal Riemann-Liouville fractional integral operator of order α.

The above-mentioned problem has many physical applications in many areas such as heterogeneous flows and materials, turbulence, viscoelasticity and electromagnetic theory. Particularly when $\alpha = 1$, the problem (1.1) reduces to the classical Navier-Stokes equations, numerical approximations of which have been studied by many authors [1–4, 8–19, 25, 32, 34, 37–44, 46, 47]. However, for the fractional Navier-Stokes equations (FNSE) which are nonlinear in character, most of them do not have exact analytical solutions. It is shown that very few cases in which the exact solution of fractional Navier-Stokes equations can be obtained, where it have to make certain assumptions about the state of the fluid and a simple configuration for the flow pattern is to be considered. Hence it is necessary to analyze and study the approximation and numerical techniques of FNSE. However, to our best knowledge, numerical analysis of such problem for fractional Navier-Stokes equations is missing except [27, 56] in the literature. Therefore, this article aims to fill the gap, study and obtain the strong convergence approximations of FNSE like (1.1).

In recent years, there have been numerous studies on fractional diffusion equation. Lin and Xu [31] have proposed the finite difference scheme in time and Legendre spectral methods in space for the time-fractional diffusion equation. Deng [6] has established the stability and error estimates for the time fractional Fokker-Planck equation and then proved that the convergent order is $O(k^{2\alpha}+h^{\mu})$, where k is the time step size and h is the space step size. Liu et al. [35] have developed a two-grid algorithm based on the mixed finite element method for a nonlinear fourth-order reaction-diffusion equation with the time-fractional derivative of Caputo-type. Jin et al. [21], by using piecewise linear functions, have studied two semidiscrete approximation schemes, i.e., Galerkin finite element method and lumped mass Galerkin finite element method, for the homogeneous time-fractional diffusion equation. Zeng et al. [51] have studied the second-order accurate schemes for the time-fractional diffusion equation with unconditional stability based on finite element method in space and the fractional linear multistep methods in time. Besides, some other interesting works in this aspect can be found in [5, 7, 20, 22–24, 30, 33, 36, 45, 52–55, 57].

In this article, our goal is to give some detailed numerical analysis of the mixed finite element method for the problem (1.1). On one hand, the discretization in space is done by the mixed finite element method. First of all, the velocity is split into two parts by introducing a linearized discrete problem with solution v_h. In particular, Motivated by the Ritz-Volterra projection, we then introduce the fractional Stokes-Volterra projection $S_h u$, the role of which is similar to that of a Ritz projection in treating the heat equation. Subsequently, with virtue of the property of the operator E_h as well as the standard duality arguments, the L^2-error estimate for the velocity is shown. On the other hand, firstly following the idea of Zhuang et al. [53] that has discretized the Riemann-Liouville fractional derivative $\mathscr{B}^\alpha$ in time, then we adopt the finite difference method and obtain the stability and convergence properties related to the time discretization. The stability analyses of semi-discretization scheme and fully discrete scheme are discussed in detail. Furthermore, We give the convergence analysis for both semidiscrete and fully discrete schemes and prove that the numerical solution converges the exact one with order $O(h^2 + k)$, where h and k respectively the space step size and the time step size.

The structure of this paper is as follows: In section 2, we introduce some preliminaries and notations, give the definition of the Mittag-Leffler function. In Section 3, we introduce the notations for finite element spatial semidiscretization, describe the semidiscrete Galerkin approximations about space and establish the error estimate for the velocity. In Section 4, we present several lemmas which play a crucial role in the proof of the error estimate of the