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A SINGULAR PARAMETERIZED FINITE VOLUME METHOD FOR THE ADVECTION-DIFFUSION EQUATION IN IRREGULAR GEOMETRIES*

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Abstract

Solving the advection-diffusion equation in irregular geometries is of great importance for realistic simulations. To this end, we adopt multi-patch parameterizations to describe irregular geometries. Different from the classical multi-patch parameterization method, C^1-continuity is introduced in order to avoid designing interface conditions between adjacent patches. However, singularities of parameterizations can't always be avoided. Thus, in this paper, a finite volume method is proposed based on smooth multi-patch singular parameterizations. It is called a singular parameterized finite volume method. Firstly, we present a numerical scheme for pure advection equation and pure diffusion equation respectively. Secondly, numerical stability results in L^2 norm show that the numerical method is not suffered from the singularities. Thirdly, the numerical method has second order accurate in L^2 norm. Finally, three numerical tests in different irregular geometries are presented to show efficiency of this numerical method.

Mathematics subject classification: 65D17, 65M08, 65M50.
Key words: Finite volume method, Smooth multi-patch singular parameterizations, The advection-diffusion equation, Irregular geometries.

1. Introduction

In this paper, we consider solving the advection-diffusion equation in two dimensional irregular geometries. On the one hand, the advection-diffusion equation is very useful in model transport, dispersion, diffusion or intrusion in various media (see [12, 17] and references listed therein). By this way, it can be used to solve problems concerning dispersion in porous media [14], the chemo-taxis in biology [20], dispersion of contaminants in rivers, lakes, embouchures and coasts [22], flow of solute material through a tube [18], the transportation of pollutants in atmosphere [30], cooling problems in generators [11], and the thermal pollution in water systems [3], etc. On the other hand, it is difficult to solve the advection-diffusion equation in irregular geometries. For instance, Darcy flows passing through a prous medium with embedded rocks can be modeled by the advection-diffusion equation [2]. Embedded rocks usually have complicated geometries and specific boundary conditions which are defined on these rocks' borders.

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Numerically, in order to solve partial differential equations (PDEs) in an irregular geometry, we need to discretize the geometry first. Then, based on the discretization, a numerical scheme is built. For example, by the Cartesian grid methods [13,21] to discretize an irregular geometry, numerical schemes attract a lot of researchers' attention because the grid is easy to be generated without considering irregular geometries. However, the Cartesian grid methods cannot deal with boundary conditions easily, because boundary points and grid points don't coincide. There has been considerable research on this topic, such as [1, 5, 10, 26]. Recently, solving PDEs on geometries modeled by computer-aided design (CAD), some studies [7,19] are reported on finite volume methods by parameterizations. In [7], Heinrich *et al.* applied the finite volume method on NURBS geometries in fluid-structure interaction. In [19], Pantaleón compares the results by Isogeometric analysis [8] with the ones by the finite volume method on NURBS geometries in different applications including fluid flow and heat transfer. These methods can be used to solve PDEs on NURBS geometries without geometric discretization errors because these CAD-geometries are exactly represented by NURBS parameterization. Moreover, we notice that in [7, 19] the geometries are discretized by single-patch parameterizations. However, restricted by the topologies of parameter domains (*e.g.* rectangles) of splines, an irregular geometry cannot usually be parameterized by just a single spline patch. Hence, multi-patch case should be considered. To solve the compatibility of advection-diffusion equation on different patches, the design of interface conditions [31, 32] needs to be considered if there is no smoothness requirement between adjacent patches. In this paper, parameterizations called smooth multi-patch parameterizations satisfy C^1-continuity between adjacent patches. Based on this smooth requirement, the design of interface conditions along common edges between adjacent patches can be avoided.

Based on recent results on smooth multi-patch parameterization, a deeper understanding of their properties are presented. For instance, singularities of this type of parameterizations can't always be avoided because of the topology of parametric meshes or the geometries of physical domains. [28, 29] prove that singularities have to appear at extraordinary vertices, i.e., these isolated singularities result from the topology of the underlaying parametric mesh. Moreover, in [16], there must be singularities to describe a circle even if it is described by a single-patch parameterization. These isolated singularities result from the geometry of circles. Thus unlike the finite volume method in previous works [7, 19], we analyse this parameterized finite volume method by considering the existence of isolated singularities. Moreover, different from designing of interface conditions [31, 32], a new challenge arises if a smooth multi-patch parameterization is used to describe a given irregular geometry. Because isolated singularities may bring numerical instability to a numerical scheme. The contributions of this work are listed in the following:

1. Although there are isolated singularities, the parameterized finite volume method proposed in this paper is numerically stable and it is a second order accurate method for the advection-diffusion equation in irregular geometries.

2. In additional, the properties of the parameterized finite volume method for the advection-diffusion equation are discussed, such as CFL condition and the non-negativity property. Especially, the choosing of time steps is discussed under the parameterization with isolated singularities.

3. At last, three typical numerical experiments are presented to show efficiency of this numerical method. The parameterizations related to these experiments are of different types